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LABORATORY GUIDE

Linear Double Inverted Pendulum Experiment for LabVIEW™ Users

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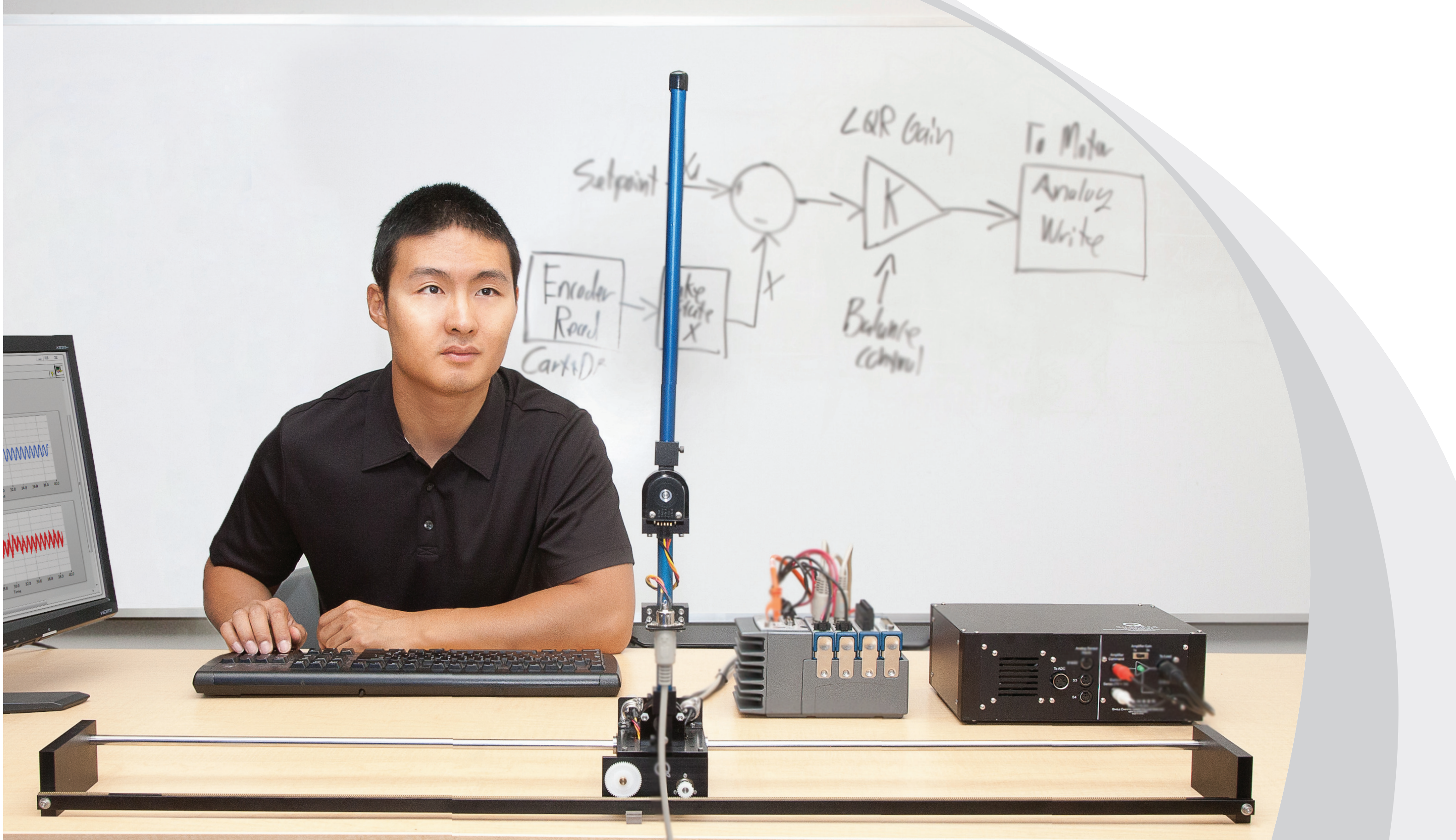

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1 INTRODUCTION

This laboratory manual describes how to design a state-feedback control system to balance a double-pendulum, while tracking a desired IP02 linear servo cart position.

The plant has two main components: the Quanser IP02 linear motion plant and the Quanser Linear Double Pendulum module. The double pendulum is a short pendulum with an additional encoder that is mounted between the IP02 linear cart, and a medium length pendulum. The medium pendulum is described in the SPG and SIP User Manual [2].

Topics Covered

- Obtain a state-space representation of the open-loop system.
- Design and tune an LQR-based state-feedback controller satisfying the closed-loop system's desired design specifications.
- Simulate the system and ensure it is stabilized using the designed state-feedback control.
- Implement the state-feedback controller on the Linear Double Pendulum (DBPEN-LIN) system and evaluate its actual performance.

Prerequisites
In order to successfully carry out this laboratory, the user should be familiar with the following:

1. See the system requirements in Section 4 for the required hardware and software.
2. Modeling and state-space representation.
3. State-feedback design using Linear-Quadratic Regular (LQR) optimization.
4. Basics of LabVIEW™ .
5. LabVIEW Integration lab detailed in Appendix A in the IP02 Laboratory Workbook [4].

2 BACKGROUND

2.1 Modeling

2.1.1 Model Convention

The double pendulum model is shown in Figure 2.1. The DBPEN-LIN is attached to the IP02 Linear Servo Base Unit pendulum pivot. The positive sense of rotation is defined to be counter-clockwise (CCW), when facing the linear cart pinions. The positive direction of linear displacement of the IP02 cart is to the right when facing the cart. Finally, the zero angle of the pendulums, θ = 0 and α = 0, corresponds to the two pendulums perfectly balanced vertically.

The IP02 cart position is denoted by the variable x_c, has a mass of m_c, and is actuated by an applied force F_c. The pendulums have masses m_{p1} and m_{p2} located at their respective centres of mass, (x_{p1}, y_{p1}) and (x_{p2}, y_{p2}). The angle of the first pendulum, α, is defined relative to the upright position while the angle of the second pendulum is defined relative to the first pendulum. The length of the distance between the first pivot and the first pendulum centre of mass is defined as l_{p1}, while the length between the second pivot and the second pendulum centre of mass is l_{p2}.

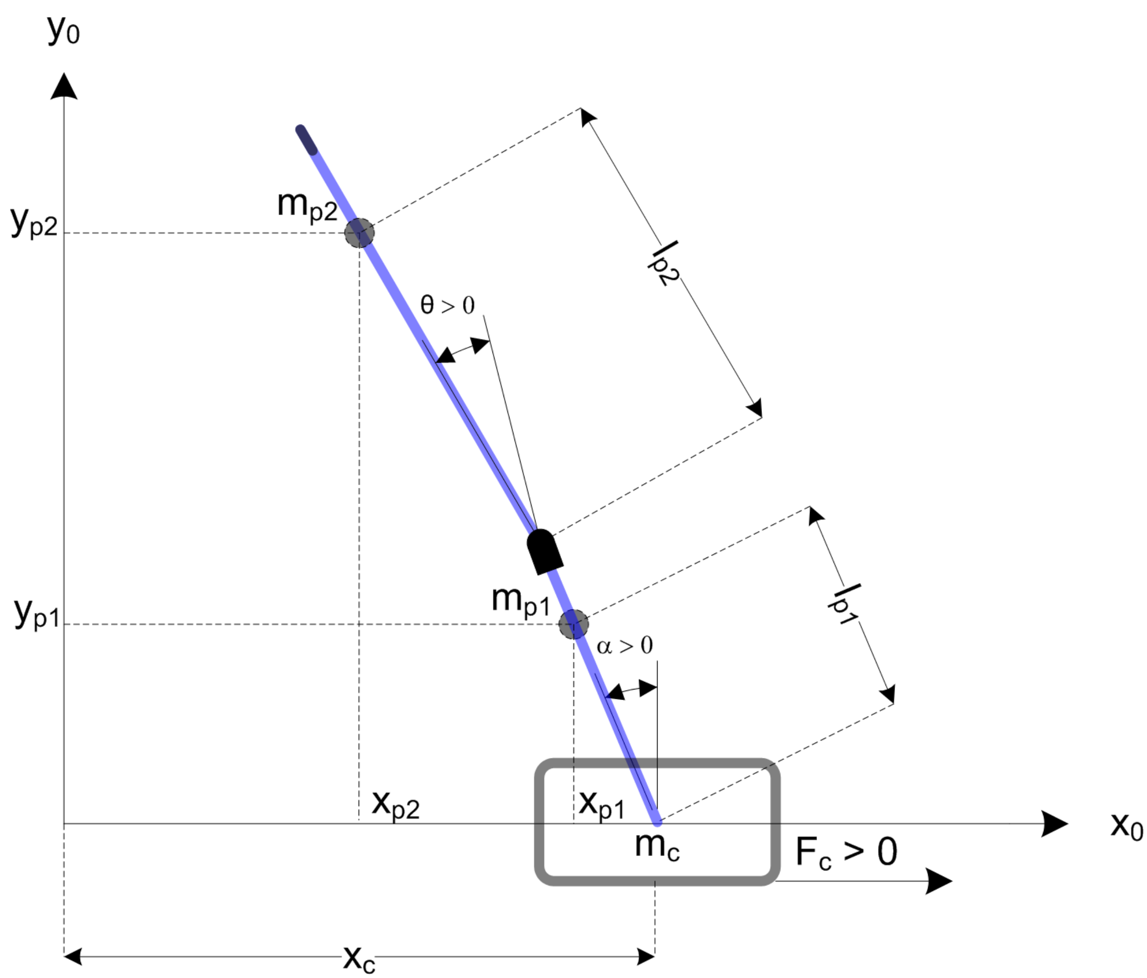


Figure 2.1: Flexible pendulum conventions

2.1.2 Nonlinear Equations of Motion

Instead of using classical mechanics, the Lagrange method is used to find the equations of motion of the system. This systematic method is often used for more complicated systems such as robot manipulators with multiple joints.

More specifically, the equations that describe the motions of the IP02 cart and pendulums with respect to the servo motor voltage, i.e. the dynamics, will be obtained using the Euler-Lagrange equation:

$$\frac{\partial^2 L}{\partial t \partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i$$

The variables q_i are called *generalized coordinates*. For this system let

$$q(t)^\top = \begin{bmatrix} x_c(t) & \alpha(t) & \theta(t) \end{bmatrix}$$

where, as shown in Figure 2.1 $\alpha(t)$ is the first pendulum angle, θ is the second pendulum angle, and $x_c(t)$ is the linear cart position. The corresponding velocities are

$$\dot{q}(t)^\top = \begin{bmatrix} \frac{\partial x_c(t)}{\partial t} & \frac{\partial \alpha(t)}{\partial t} & \frac{\partial \theta(t)}{\partial t} \end{bmatrix}$$

Note: The dot convention for the time derivative will be used throughout this document, e.g., $\dot{\alpha} = \frac{d\alpha}{dt}$. The time variable t will also be dropped from α, θ, and x_c, e.g., $\alpha = \alpha(t)$.

With the generalized coordinates defined, the Euler-Lagrange equations for the rotary pendulum system are

$$\begin{aligned} \frac{\partial^2 L}{\partial t \partial \dot{x_c}} - \frac{\partial L}{\partial x_c} &= Q_1 \\ \frac{\partial^2 L}{\partial t \partial \dot{\alpha}} - \frac{\partial L}{\partial \alpha} &= Q_2 \\ \frac{\partial^2 L}{\partial t \partial \dot{\theta}} - \frac{\partial L}{\partial \theta} &= Q_2 \end{aligned}$$

The Lagrangian of the system is described

$$L = T - V$$

where T is the total kinetic energy of the system and V is the total potential energy of the system. Thus the Lagrangian is the difference between a system's kinetic and potential energies.

The generalized forces Q_i are used to describe the non-conservative forces (e.g., friction) applied to a system with respect to the generalized coordinates. In this case the Coulomb friction is neglected, as well as the force due to the pendulum acting on the linear cart. The generalized forces acting on the system are thus:

$$Q_1 = F_c - B_{eq}\dot{x_c}, \tag{2.1}$$

$$Q_2 = -B_{p1}\dot{\alpha}, \tag{2.2}$$

and

$$Q_3 = -B_{p2}\dot{\theta}. \tag{2.3}$$

Our control variable is the input servo motor voltage, V_m.

The force applied to the linear cart, F_c, is generated by the servo motor as described by the equation

$$F_c = \frac{\eta_g K_g K_t}{R_m r_{mp}} \left(-\frac{K_g K_m \dot{x_c}}{r_{mp}} + \eta_m V_m \right) \tag{2.4}$$

The Euler-Lagrange equations is a systematic method of finding the equations of motion, i.e., EOMs, of a system. Once the kinetic and potential energy are obtained and the Lagrangian is found, then the task is to compute various derivatives to get the EOMs. After going through this process, the nonlinear equations of motion for the system can be obtained. **See the supplied Maple worksheet (or its equivalent HTML representation) for the complete derivation.**

Based on the system schematic shown in Figure 2.1 and the generalized forces Equation 2.1, Equation 2.2 and Equation 2.2, the first Lagrange equation can be expressed as shown in the the *DBPEN-LIN.mws* Maple^{TM} worksheet, or the HTML equivalent. The Maple^{TM} worksheet also details the non-linear equations of motion that result from solving the two Lagrange equations for the second-order time derivative of the Lagrangian coordinates.

See [1] for a description of the corresponding IP02 parameters (e.g. such as the back-emf constant, K_m).

2.1.3 Linearizing

Here is an example of how to linearize a two-variable nonlinear function called $f(z)$. Variable z is defined

$$z^\top = [z_1 \; z_2]$$

and $f(z)$ is to be linearized about the operating point

$$z_0{}^\top = [a \; b]$$

The linearized function is

$$f_{lin} = f(z_0) + \left(\frac{\partial f(z)}{\partial z_1}\right)\Bigg|_{z=z_0}(z_1 - a) + \left(\frac{\partial f(z)}{\partial z_2}\right)\Bigg|_{z=z_0}(z_2 - b)$$

2.1.4 Linear State-Space Model

The linear state-space equations are

$$\dot{x} = Ax + Bu \tag{2.5}$$

and

$$y = Cx + Du \tag{2.6}$$

where x is the state, u is the control input, A, B, C, and D are state-space matrices. For the linear double pendulum system, the state and output are defined

$$x^\top = \begin{bmatrix} x_c & \alpha & \gamma & \dot{x_c} & \dot{\alpha} & \dot{\gamma} \end{bmatrix}$$

where $x_{c,d}$ is the desired cart position, and

$$y^\top = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}.$$

After linearizing the nonlinear equations of motion about the zero angle (balanced) position, and substituting the state given in Equation 2.1.4, we obtain the state-space matrices presented in the *DBPEN-LIN.mws* Maple^{TM} worksheet, or the HTML equivalent.

Note: The velocities of the servo and pendulum angles can be computed in the digital controller, e.g., by taking the derivative and filtering the result though a high-pass filter.

2.2 Control

In Section 2.1, we found a linear state-state space model that represents the Linear Double Pendulum system. This model is used to investigate the stability properties of the system in Section 2.2.1. In Section 2.2.2, the notion of controllability is introduced. Using the Linear Quadratic Regular algorithm, or LQR, is a common way to find the control gain and is discussed in Section 2.2.3. Lastly, Section 2.2.4 describes the state-feedback control used to control the servo position while minimizing link deflection.

2.2.1 Stability

The stability of a system can be determined from its poles ([5]):

- Stable systems have poles only in the left-hand plane.
- Unstable systems have at least one pole in the right-hand plane and/or poles of multiplicity greater than 1 on the imaginary axis.
- Marginally stable systems have one pole on the imaginary axis and the other poles in the left-hand plane.

The poles are the roots of the system's characteristic equation. From the state-space, the characteristic equation of the system can be found using

$$\det(sI - A) = 0 \tag{2.7}$$

where *det()* is the determinant function, s is the Laplace operator, and I the identity matrix. These are the *eigenvalues* of the state-space matrix A.

2.2.2 Controllability

If the control input, u, of a system can take each state variable, x_i where $i = 1 \dots n$, from an initial state to a final state then the system is controllable, otherwise it is uncontrollable ([5]).

Rank Test The system is controllable if the rank of its controllability matrix

$$T = \begin{bmatrix} B \; AB \; A^2B \dots A^nB \end{bmatrix} \tag{2.8}$$

equals the number of states in the system,

$$\text{rank}(T) = n. \tag{2.9}$$

2.2.3 Linear Quadratic Regular (LQR)

If (A,B) are controllable, then the Linear Quadratic Regular optimization method can be used to find a feedback control gain. Given the plant model in Equation 2.5, find a control input u that minimizes the cost function

$$J = \int_0^\infty x(t)'Qx(t) + u(t)'Ru(t)\,dt, \tag{2.10}$$

where Q and R are the weighting matrices. The weighting matrices affect how LQR minimizes the function and are, essentially, tuning variables.

Given the control law $u = -Kx$, the state-space in Equation 2.5 becomes

$$\begin{aligned} \dot{x} &= Ax + B(-Kx) \\ &= (A - BK)x \end{aligned}$$

2.2.4 Feedback Control

The feedback control loop that in Figure 2.2 is designed to control the position of the IP02 linear cart while balancing the rigid and flexible pendulums.

The reference state is defined

$$x_d = \begin{bmatrix} x_{c,d} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The controller is therefore

$$u = K(x_d - x). \tag{2.11}$$

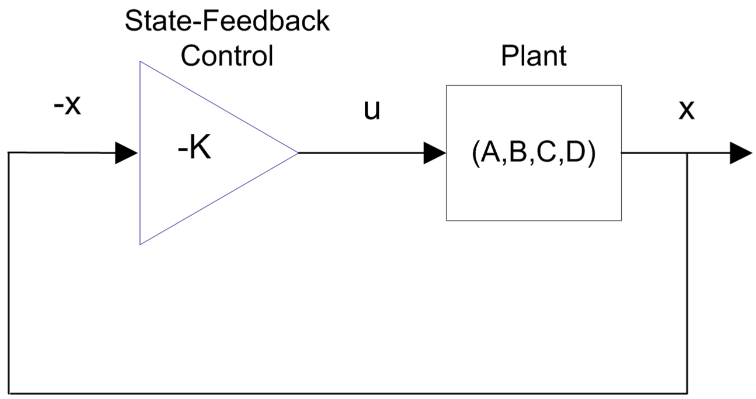


Figure 2.2: State-feedback control loop

Note that when $x_d = 0$, then $u = -Kx$, which is the controller used in the LQR algorithm.

To eliminate linear cart regulation error, we can augment the system to include an integrator such that

$$\dot{\eta} = \begin{bmatrix} A & 0 \\ 1 & 0 \end{bmatrix} \eta + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$

where A and B are the state-space matrices defined in Section 2.1.4 and the states are

$$\eta^\top = \begin{bmatrix} x_c - x_{c,d} & \alpha & \theta & \dot{x_c} & \dot{\alpha} & \dot{\theta} & \int (x_c - x_{c,d})\, \mathrm{d}t \end{bmatrix}$$

This introduces the integration terms $\eta_7(t) = \int (x_c - x_{c,d})\, \mathrm{d}t$ to the feedback controller

$$u = -K(\eta),$$

to help compensate for unmodeled dynamics in the actual system that cause the cart to drift from its desired postiion.

3 LAB EXPERIMENTS

3.1 Simulation

In this section we will use the VI shown in Figure 3.1 to simulate the closed-loop control of the Linear Double Pendulum system. The system is simulated using the linear model summarized in Section 2.1. The VI uses the state-feedback control described in Section 2.2.4. The feedback gain K is found using the LQR command from the *Control Design and Simulation Toolkit* (LQR is described briefly in Section 2.2.3). The goal is to make sure the gain used successfully stabilizes the system (i.e., keeps the pendulums balanced), and does not saturate the dc motor.

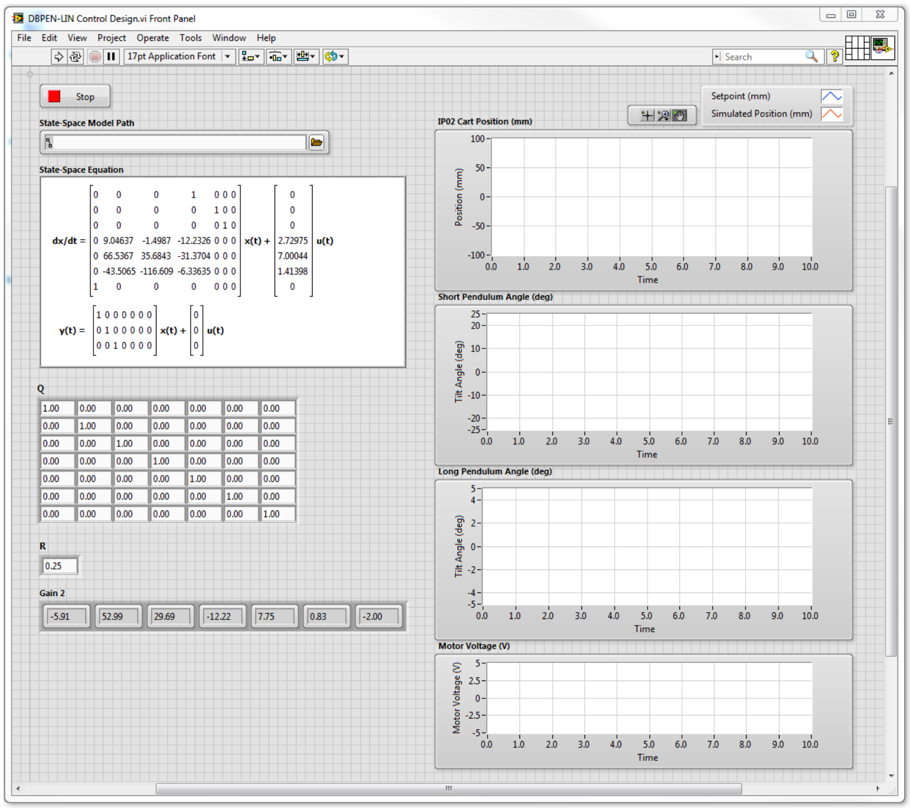


Figure 3.1: VI used to simulate Linear Double Pendulum.

IMPORTANT: Before you can conduct these experiments, you need to make sure that the lab files are configured according to your setup. If they have not been configured already, then you need to go to Section 4 to configure the lab files first.

3.1.1 Procedure

Follow these steps to simulate the system:

1. Open and run *DBPEN-LIN Control Design.vi* as described in Section 4. Make sure you choose your model file using the *Model Path* control. The model file is generated using the *DBPEN-LIN Modeling* VI by entering the state space model and exporting the resultant model file.

2. By default, the Q matrix is sent to identity matrix. Set the LQR weighting matrices to

$$Q = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 50 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 50 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

   and

$$R = 0.01.$$

3. This automatically generates the gain

$$K = \begin{bmatrix} 42.33 & -180.91 & -417.89 & 31.06 & -48.50 & -43.24 & 10 \end{bmatrix}.$$

   LQR Tuning: When tuning the LQR, we start with the identity matrix. To put more emphasis on the response of the linear cart, we set the cart position gain $Q(1,1)$ = 10, and to ensure that the pendulums remain balanced we set $Q(2,2)$ = 50 and $Q(3,3)$ = 50. The pendulum damping terms are set to 0.1 to add some slight damping to the response of the pendulum angles, while maintaining a fast response. The last diagonal element, $Q(7,7)$ is set to 1 to generate an integral gain for the linear cart to keep it tracking the desired position on the track.

4. Run the VI. The scopes should be displaying responses similar to Figure 3.2.

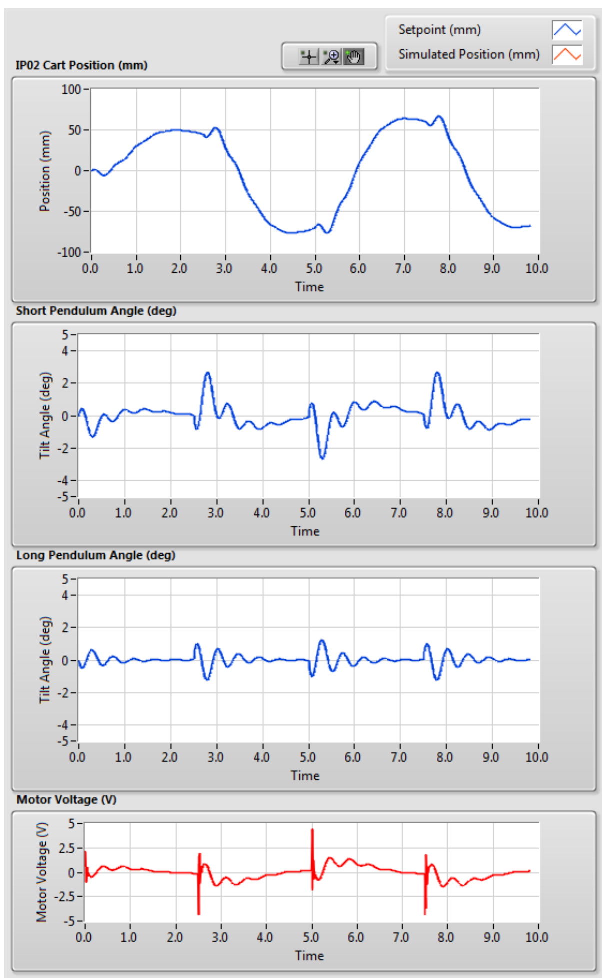


Figure 3.2: Simulated closed-loop response.

5. Click on the *STOP* button to stop running the VI.

3.1.2 Analysis

As shown by the response in Figure 3.2, the pendulums maintain their balanced positions despite the applied 50 mm cart position disturbance. Further analysis can be performed using the *Graph Palette*.

3.2 Implementation

The *DBPEN-LIN Balance Control* VI shown in Figure 3.3 is used to perform the balance control on the DBPEN-LIN. The VI contains Quanser Rapid Control Prototyping Toolkit®blocks that interface with the dc motor and sensors of the DBPEN-LIN system.

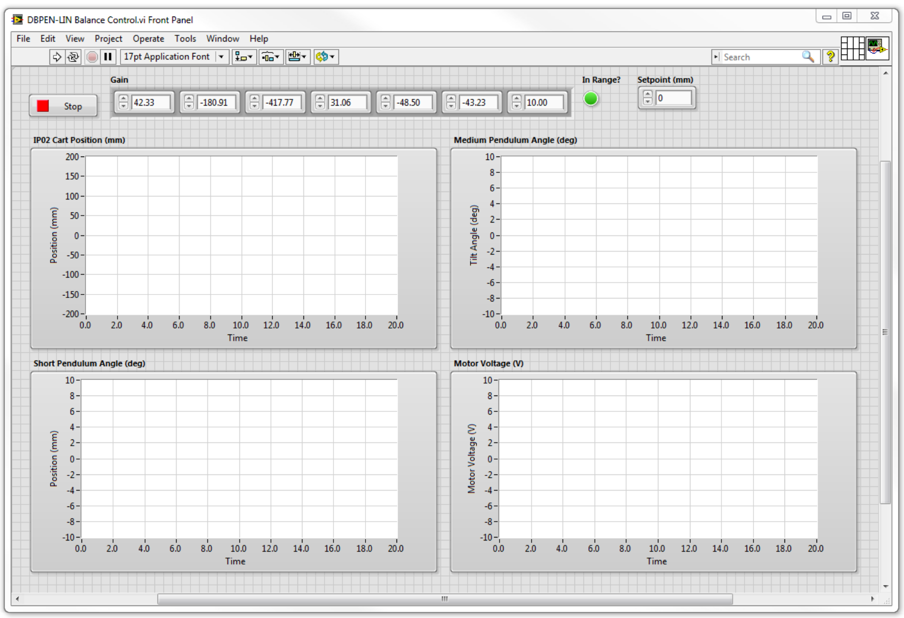


Figure 3.3: VI used to run controller on the DBPEN-LIN.

IMPORTANT: Before you can conduct these experiments, you need to make sure that the lab files are configured according to your setup. If they have not been configured already, then you need to go to Section 4 to configure the lab files first.

3.2.1 Procedure

Follow this procedure:

1. Make sure gain K is set to the gain you found and simulated in Section 3.1.
2. **Make sure that the cart is close to the centre of the track and the pendulums are completely still.**
3. Run the VI.
4. Once the controller is running, slowly raise the pendulums counter-clockwise (CCW) to their upright vertical position. You should feel the motor voltage kick-in when the pendulums are perfectly straight, and the balance control engages. Holding the tip of the medium pendulum at 90 degrees is an effective way of ensuring that both pendulums are perfectly straight. The scopes should be displaying responses similar to Figure 3.4.

Note: Once the controller has engaged, do not attempt to manually lower the pendulums. If the pendulums or cart move outside of a safe workspace, the system watchdog should halt the controller automatically.

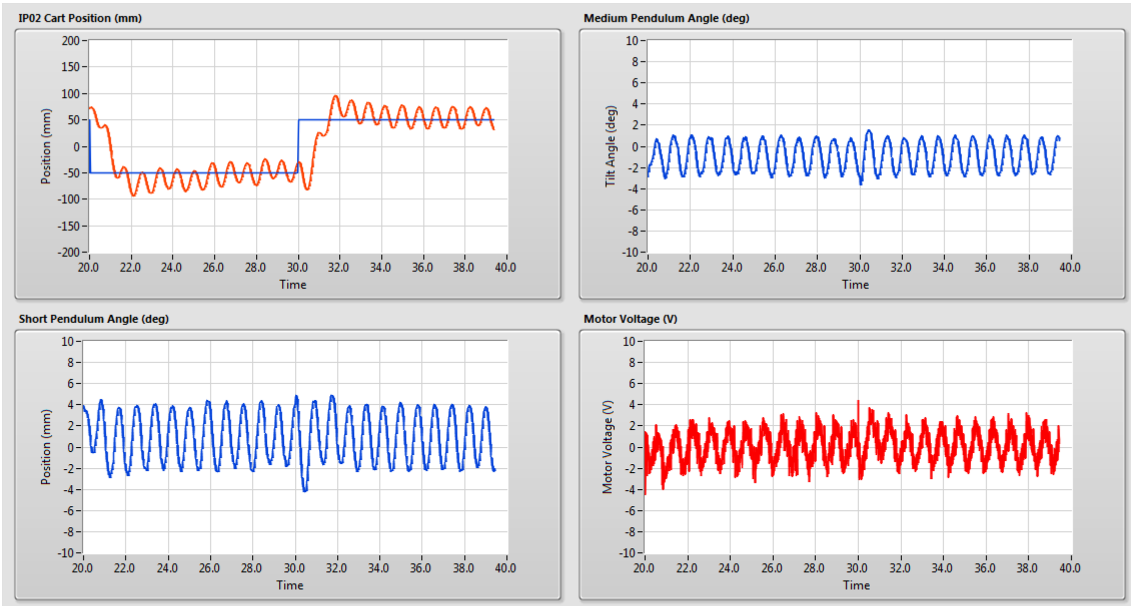


Figure 3.4: Typical response when balancing the DBPEN-LIN system

5. To stop the experiment, click on the *Stop* button but **make sure you catch the pendulums before they swing down**.

3.2.2 Analysis

An example of the balance control response is shown in Figure 3.5.

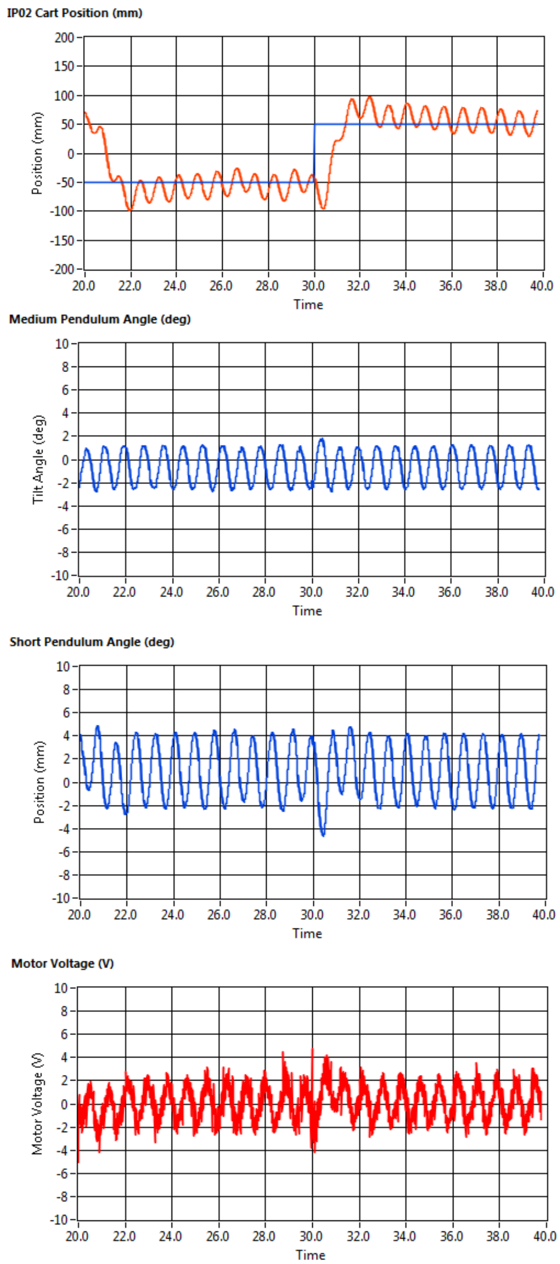


Figure 3.5: DBPEN-LIN balance control response

Due to the friction and other non-linearities in the system, the IP02 servo oscillates back-and-forth approximately $\pm$40 mm to balance the pendulum. The small pendulum angle does not exceed 5 degrees and the medium pendulum angle does not exceed 3 degrees when balanced. Because of the integrator, the IP02 cart eventually settles at the desired position.

4 SYSTEM REQUIREMENTS

Required Software

Make sure LabVIEW™ is installed with the following required add-ons:

1. LabVIEW™
2. NI-DAQmx
3. NI LabVIEW™ Control Design and Simulation Module
4. NI LabVIEW™ MathScript RT Module
5. Quanser Rapid Control Prototyping Toolkit®

Note: Make sure the Quanser Rapid Control Prototyping (RCP) Toolkit is installed after LabVIEW. See the RCP Toolkit Quick Start Guide for more information.

Required Hardware

- Data acquisition (DAQ) device **with 2x encoder inputs** and that is compatible with Quanser Rapid Control Prototyping Toolkit®.
- Quanser IP02 linear servo.
- Quanser Linear Double Pendulum (positioned underneath the IP02).
- Quanser VoltPAQ-X1 power amplifier, or equivalent.

Before Starting Lab

Before you begin this laboratory make sure:

- LabVIEW™ is installed on your PC.
- DAQ device has been successfully tested (e.g., using the test software in the Quick Start Guide or the *Analog Loopback Demo*).
- Linear Double Pendulum and amplifier are connected to your DAQ board as described its User Manual [3].

4.1 Overview of Files

File Name	Description
Linear Double Pendulum User Manual.pdf	This manual describes the hardware of the DBPEN-LIN system and explains how to setup and wire the system for the experiments.
Linear Double Pendulum Laboratory Manual.pdf	This document demonstrates how to obtain the linear state-space model of the system, simulate the closed-loop system, and implement controllers on the DBPEN-LIN plant using LabVIEW™ .
DBPEN-LIN Project.lvproj	LabVIEW project that contains all the VIs required for the lab.
DBPEN-LIN Modeling.vi	VI used to generate the linear state-space model of the DBPEN-LIN system.
DBPEN-LIN Control Design.vi	VI used to design the LQR state-feedback gain and simulate the DBPEN-LIN system.
DBPEN-LIN Balance Control.vi	VI that implements the state-feedback control on the DBPEN-LIN system.
DBPEN-LIN.mws	Maple worksheet used to develop the model for the DBPEN-LIN experiment. Waterloo Maple 9, or a later release, is required to open, modify, and execute this file.
DBPEN-LIN.html	HTML presentation of the Maple Worksheet. It allows users to view the content of the Maple file without having Maple 9 installed. No modifications to the equations can be performed when in this format.

Table 4.1: Files supplied with the DBPEN-LIN

4.2 Setup for Simulation

Before beginning the in-lab procedure outlined in Section 3.1, the modeling and control design VIs must must be configured.

Follow these steps:

1. Load LabVIEW™ .
2. Open the *DBPEN-LIN Project.lvproj* LabVIEW project, shown in Figure 4.1.
3. Open the *DBPEN-LIN Modeling.vi* shown in Figure 4.2.
4. The pendulum and IP02 parameters are already set, by default. Run the VI to generate the linear state-space model.
5. In *Model Name*, enter the name of the model you and click on OK. This will save the state-space model under the folder *Model Files*. You can close this VI now.
6. Open the *DBPEN-LIN Control Design* VI, shown in Figure 3.1.
7. Using the *File Path* control, select the model file.
8. Run the VI. The state-space model should load. You are now ready to design your LQR control and simulate the closed-loop response.

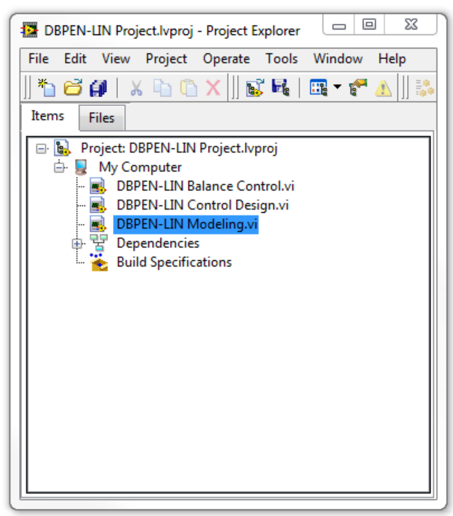


Figure 4.1: LabVIEW Linear Double Pendulum Project

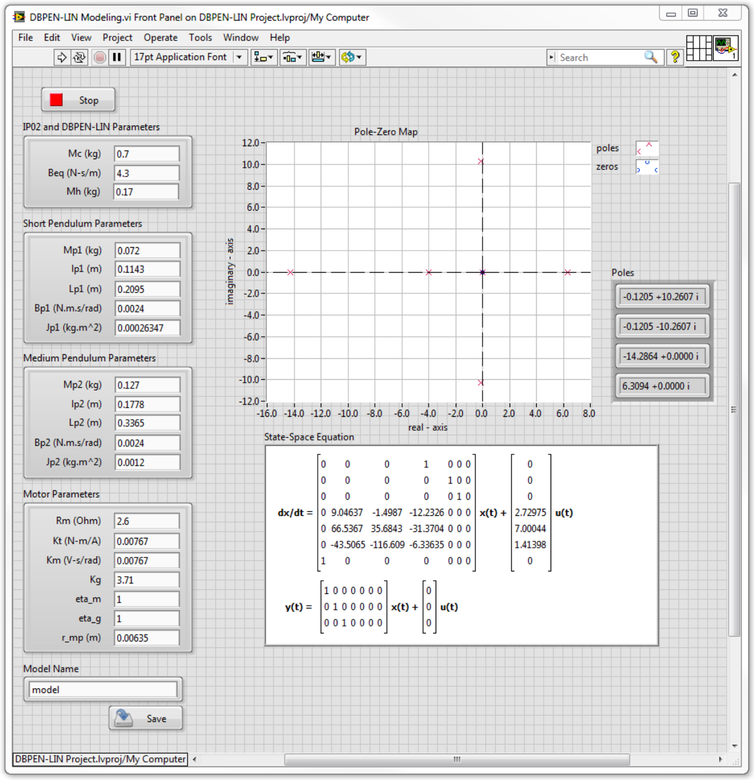


Figure 4.2: Linear Double Pendulum Modeling VI

4.3 Setup for Running on DBPEN-LIN

Before performing the in-lab exercises in Section 3.2, the DBPEN-LIN system and the *DBPEN-LIN Balance Control.vi* must be configured properly.

Follow these steps to get the system ready for this lab:

1. Setup the IP02 with the DBPEN-LIN module as detailed in the Linear Double Pendulum User Manual [3].
2. **Make sure that the cart is close to the centre of the track and the pendulums are completely still.**. For more information, go to the DBPEN-LIN User Manual [3].
3. Open the *DBPEN-LIN Balance Control.vi*, shown in Figure 3.3.
4. Set gain K control in the VI to the value found in Section 4.2 (or another gain you want to test on the system).
5. **Configure DAQ:** Ensure the HIL Initialize block is configured for the DAQ device that is installed in your system. To do this, go to the block diagram (CTRL-E) and double click on the **HIL Initialize** Express VI shown in Figure 4.3.

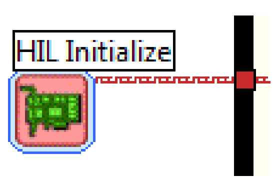


Figure 4.3: HIL Initialize Express VI

6. Under the Main tab, select the data acquisition device that is installed on your system in the *Board type* section. For example, in Figure 4.4 the Q8-USB is chosen.

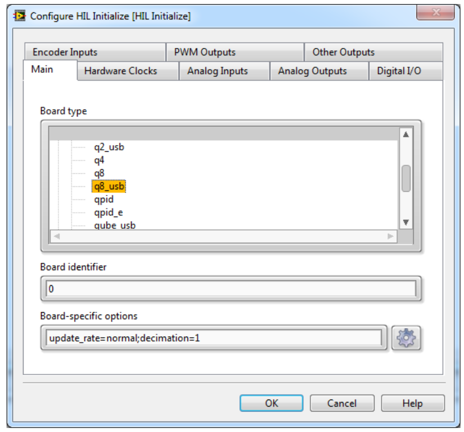


Figure 4.4: Select DAQ board that will be used to control system

REFERENCES

[1] Quanser Inc. *IP02 User Manual*, 2009.

[2] Quanser Inc. *SIP and SPG User Manual*, 2009.

[3] Quanser Inc. *DBPEN-LIN User Manual*, 2012.

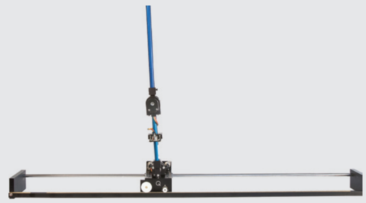
[4] Quanser Inc. *IP02 Lab Workbook (LabVIEW)*, 2012.

[5] Norman S. Nise. *Control Systems Engineering*. John Wiley & Sons, Inc., 2008.

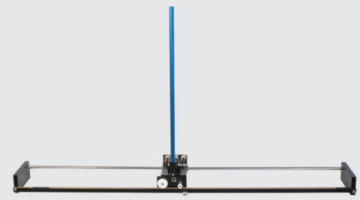
Nine linear motion plants for teaching fundamental and advanced controls concepts

IP02 Base Unit

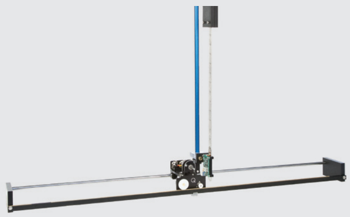
Linear Double Inverted Pendulum



Linear Pendulum

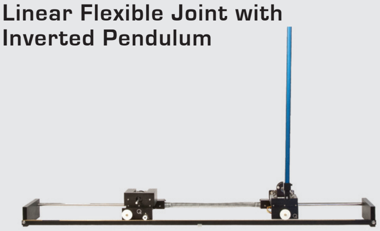


Linear Flexible Inverted Pendulum

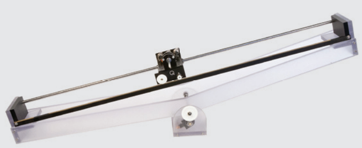


Linear Flexible Joint

Linear Flexible Joint with Inverted Pendulum

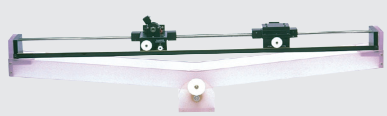


Seesaw



Seesaw Pendulum

Linear Flexible Joint on Seesaw



Quanser’s linear collection allows you to create experiments of varying complexity – from basic to advanced. With nine plants to choose from, students can be exposed to a wide range of topics relating to mechanical and aerospace engineering. For more information please contact info@quanser.com

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