



# STUDENT WORKBOOK

## Flexible Link Experiment for LabVIEW™ Users

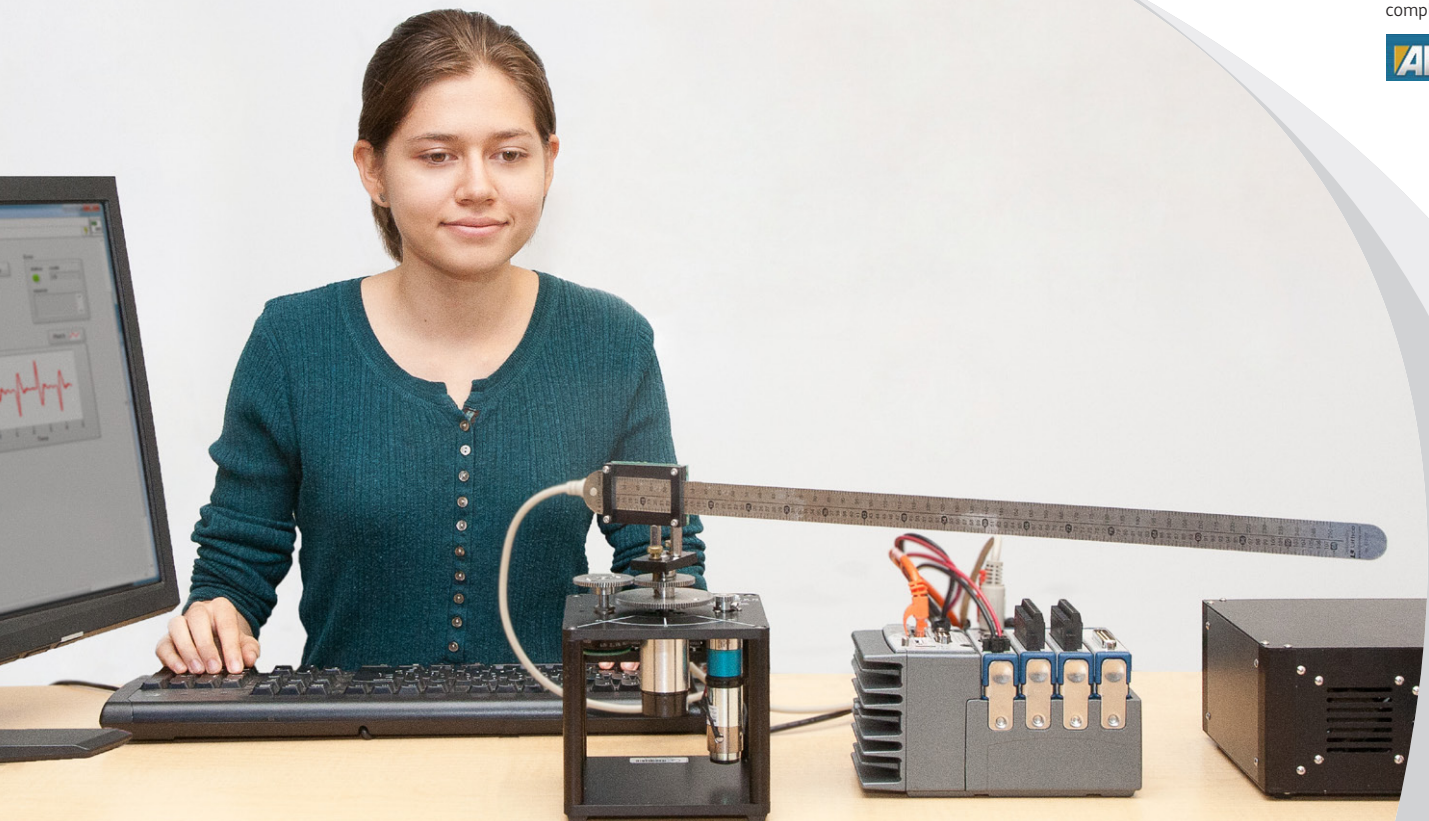
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# 1 INTRODUCTION

The objective of this experiment is to control the position of the servo while minimizing the motions the flexible rotary link.

## Topics Covered

- Modeling the Rotary Flexible Link using Lagrange.
- Find the linear state-space model of the system.
- Do some basic model validation.
- Design a state-feedback controller using Linear-Quadratic Regulator (LQR) algorithm through simulation.
- Implement the designed LQR controller on the device.
- Compare the simulated and measured closed-loop results.
- Assess the behaviour of implementing a partial-state feedback controller.

## Prerequisites

In order to successfully carry out this laboratory, the user should be familiar with the following:

- Basics of **LabVIEW™**.
- Transfer function fundamentals.
- State-space modeling, e.g., obtaining state equations from a set of differential equations.
- *SRV02 LabVIEW Integration Laboratory* (in Appendix A of [4]) in order to be familiar using **LabVIEW™** with the SRV02.

## 2 MODELING

### 2.1 Background

#### 2.1.1 Model

The Rotary Flexible Link model is shown in Figure 2.1. The base of the flexible link is mounted on the load gear of the SRV02 system. The servo angle,  $\theta$ , increases positively when it rotates counter-clockwise (CCW). The servo (and thus the link) turn in the CCW direction when the control voltage is positive, i.e.,  $V_m > 0$ .

The flexible link has a total length of  $L_l$ , a mass of  $m_l$ , and its moment of inertia about the center of mass is  $J_l$ . See the *Rotary Flexible Link User Manual* (in [5]) for the values of these parameters. The deflection angle of the link is denoted as  $\alpha$  and increases positively when rotated CCW.

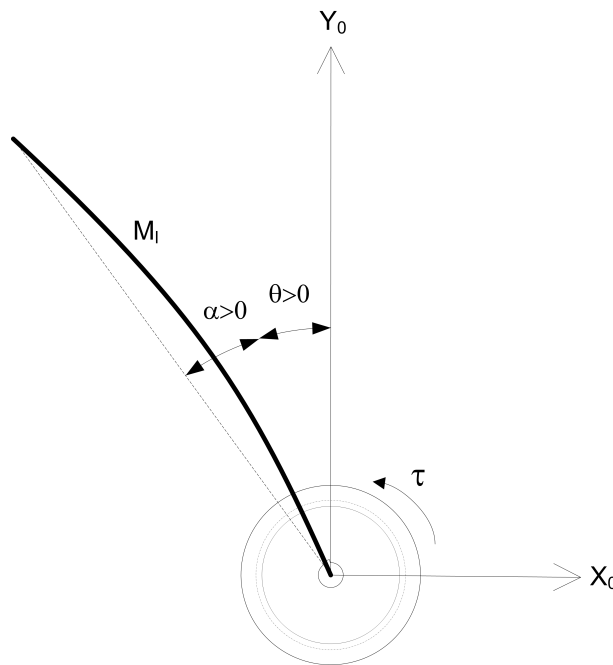


Figure 2.1: Rotary Flexible Link Angles

The flexible link system can be represented by the diagram shown in Figure 2.2. Our control variable is the input servo motor voltage,  $V_m$ . This generates a torque,  $\tau$ , at the load gear of the servo that rotates the base of the link. The viscous friction coefficient of the servo is denoted by  $B_{eq}$ . This is the friction that opposes the torque being applied at the servo load gear. The friction acting on the link is represented by the viscous damping coefficient  $B_L$ . Finally, the flexible link is modeled as a linear spring with the stiffness  $K_s$ .

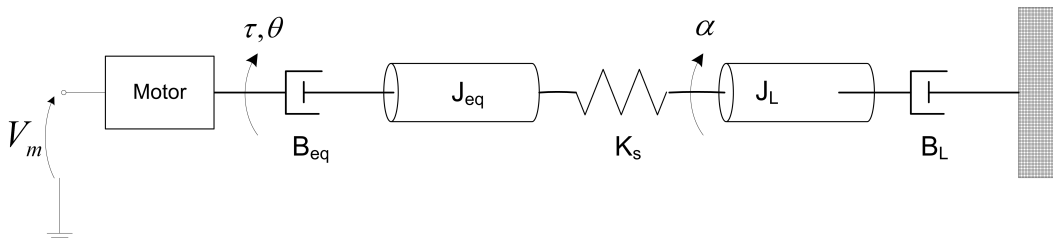


Figure 2.2: Rotary Flexible Link Model

## 2.1.2 Finding the Equations of Motion

Instead of using classical mechanics, the Lagrange method is used to find the equations of motion of the system. This systematic method is often used for more complicated systems such as robot manipulators with multiple joints.

More specifically, the equations that describe the motions of the servo and the link with respect to the servo motor voltage, i.e. the dynamics, will be obtained using the Euler-Lagrange equation:

$$\frac{\partial^2 L}{\partial t \partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i \quad (2.1)$$

The variables  $q_i$  are called *generalized coordinates*. For this system let

$$q(t)^\top = [\theta(t) \ \alpha(t)] \quad (2.2)$$

where, as shown in Figure 2.2,  $\theta(t)$  is the servo angle and  $\alpha(t)$  is the flexible link angle. The corresponding velocities are

$$\dot{q}(t)^\top = \left[ \frac{\partial \theta(t)}{\partial t} \ \frac{\partial \alpha(t)}{\partial t} \right] \quad (2.3)$$

**Note:** The dot convention for the time derivative will be used throughout this document, i.e.,  $\dot{\theta} = \frac{d\theta}{dt}$  and  $\dot{\alpha} = \frac{d\alpha}{dt}$ . The time variable  $t$  will also be dropped from  $\theta$  and  $\alpha$ , i.e.,  $\theta := \theta(t)$  and  $\alpha := \alpha(t)$ .

With the generalized coordinates defined, the Euler-Lagrange equations for the rotary flexible link system are

$$\frac{\partial^2 L}{\partial t \partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = Q_1 \quad (2.4)$$

and

$$\frac{\partial^2 L}{\partial t \partial \dot{\alpha}} - \frac{\partial L}{\partial \alpha} = Q_2 \quad (2.5)$$

The Lagrangian of a system is defined

$$L = T - V \quad (2.6)$$

where  $T$  is the total kinetic energy of the system and  $V$  is the total potential energy of the system. Thus the Lagrangian is the difference between a system's kinetic and potential energies.

The generalized forces  $Q_i$  are used to describe the non-conservative forces (e.g., friction) applied to a system with respect to the generalized coordinates. In this case, the generalized force acting on the rotary arm is

$$Q_1 = \tau - B_{eq} \dot{\theta} \quad (2.7)$$

and acting on the link is

$$Q_2 = -B_l \dot{\alpha}. \quad (2.8)$$

The torque applied at the base of the rotary arm (i.e., at the load gear) is generated by the servo motor as described by the equation

$$\tau = \frac{\eta_g K_g \eta_m k_t (V_m - K_g k_m \dot{\theta})}{R_m}. \quad (2.9)$$

See [1] for a description of the corresponding SRV02 parameters (e.g. such as the back-emf constant,  $k_m$ ). The servo damping (i.e. friction),  $B_{eq}$ , opposes the applied torque. The flexible link is not actuated, the only force acting on the link is the damping,  $B_l$ .

Again, the Euler-Lagrange equations is a systematic method of finding the equations of motion (EOMs) of a system. Once the kinetic and potential energy are obtained and the Lagrangian is found, then the task is to compute various derivatives to get the EOMs.

### 2.1.3 Potential and Kinetic Energy

#### Kinetic Energy

Translational kinetic equation is defined as

$$T = \frac{1}{2}mv^2, \quad (2.10)$$

where  $m$  is the mass of the object and  $v$  is the linear velocity.

Rotational kinetic energy is described as

$$T = \frac{1}{2}J\omega^2 \quad (2.11)$$

where  $J$  is the moment of inertia of the object and  $\omega$  is its angular rate.

#### Potential Energy

Potential energy comes in different forms. Typically in mechanical system we deal with *gravitational* and *elastic* potential energy. The *relative* gravitational potential energy of an object is

$$V_g = mg\Delta h, \quad (2.12)$$

where  $m$  is the object mass and  $\Delta h$  is the change in altitude of the object (from a reference point). The potential energy of an object that rises from the table surface (i.e., the reference) up to 0.25 meter is  $\Delta h = 0.25 - 0 = 0.25$  and the energy stored is  $V_g = 0.25mg$ .

The equation for elastic potential energy, i.e., the energy stored in a spring, is

$$V_e = \frac{1}{2}K\Delta x^2 \quad (2.13)$$

where  $K$  is the spring stiffness and  $\Delta x$  is the linear or angular change in position. If an object that is connected to a spring moves from its initial reference position to 0.1 m, then the change in displacement is  $\Delta x = 0.1 - 0 = 0.1$  and the energy stored equals  $V_e = 0.005K$ .

### 2.1.4 Linear State-Space Model

The linear state-space equations are

$$\dot{x} = Ax + Bu \quad (2.14)$$

and

$$y = Cx + Du \quad (2.15)$$

where  $x$  is the state,  $u$  is the control input,  $A$ ,  $B$ ,  $C$ , and  $D$  are state-space matrices. For the Rotary Flexible Link system, the state and output are defined

$$x^T = [\theta \ \alpha \ \dot{\theta} \ \dot{\alpha}] \quad (2.16)$$

and

$$y^T = [x_1 \ x_2]. \quad (2.17)$$

In the output equation, only the position of the servo and link angles are being measured. Based on this, the  $C$  and  $D$  matrices in the output equation are

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (2.18)$$

and

$$D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (2.19)$$

The velocities of the servo and link angles can be computed in the digital controller, e.g., by taking the derivative and filtering the result through a high-pass filter.

## 2.1.5 Free-Oscillation of a Second Order System

The free-oscillatory equation of motion of a second-order system described by

$$J\ddot{x} + B\dot{x} + Kx = 0 \quad (2.20)$$

is shown in Figure 2.3. Assuming the initial conditions  $x(0^-) = x_0$  and  $\dot{x}(0^-) = 0$ , the Laplace transform of Equation 2.20 is

$$X(s) = \frac{\frac{x_0}{J}}{s^2 + \frac{B}{J}s + \frac{K}{J}} \quad (2.21)$$

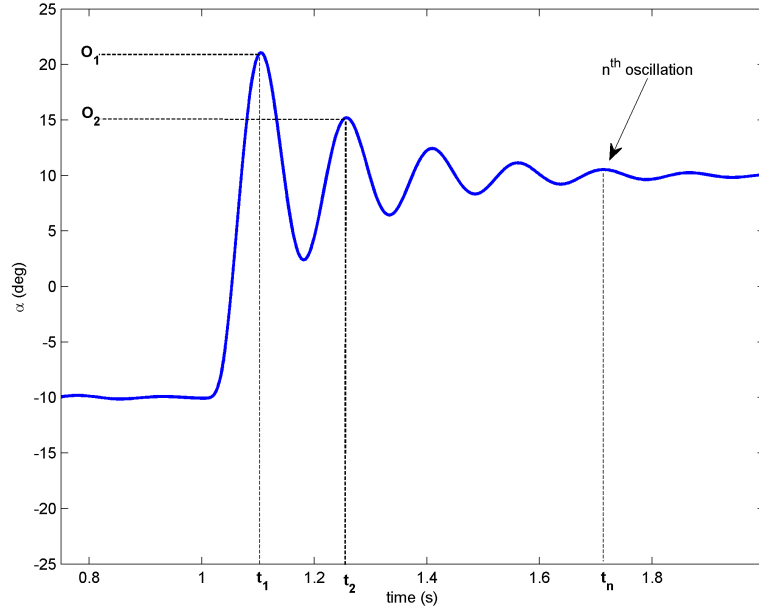


Figure 2.3: Free Oscillation Response

The prototype second-order equation is defined

$$s^2 + 2\zeta\omega_n s + \omega_n^2, \quad (2.22)$$

where  $\zeta$  is the damping ratio and  $\omega_n$  is the natural frequency. Equating the characteristic equation in 2.21 to this gives

$$\omega_n^2 = \frac{K}{J} \quad (2.23)$$

and

$$2\zeta\omega_n = \frac{B}{J} \quad (2.24)$$

Based on the measured damping ratio and natural frequency, the friction (or stiffness) of the system is

$$K = J\omega_n^2 \quad (2.25)$$

and the viscous damping is

$$B = 2\zeta\omega_n J. \quad (2.26)$$

### Finding the Natural Frequency

The period of the oscillations in a system response can be found using the equation

$$T_{osc} = \frac{t_{n+1} - t_1}{n} \quad (2.27)$$



where  $t_n$  is the time of the  $n^{\text{th}}$  oscillation,  $t_1$  is the time of the first peak, and  $n$  is the number of oscillations considered. From this, the damped natural frequency (in radians per second) is

$$\omega_d = \frac{2\pi}{T_{osc}} \quad (2.28)$$

and the undamped natural frequency is

$$\omega_n = \frac{\omega_d}{\sqrt{1 - \zeta^2}}. \quad (2.29)$$

### Finding the Damping Ratio

The damping ratio of a second-order system can be found from its response. For a typical second-order under-damped system, the subsidence ratio (i.e., decrement ratio) is defined as

$$\delta = \frac{1}{n} \ln \frac{O_1}{O_n} \quad (2.30)$$

where  $O_1$  is the peak of the first oscillation and  $O_n$  is the peak of the  $n^{\text{th}}$  oscillation. Note that  $O_1 > O_n$ , as this is a decaying response.

The damping ratio is defined

$$\zeta = \frac{1}{\sqrt{1 + \frac{2\pi^2}{\delta^2}}} \quad (2.31)$$

## 2.2 Pre-Lab Questions

1. Energy is stored in the flexible link, i.e., the "spring", as it *flexes* by an angle of  $\alpha$  (see Figure 2.1). Find the potential energy of the flexible link. Use the parameters shown in Figure 2.2.
2. Find the total kinetic energy of the system contributed by the rotary servo,  $\theta$ , and the deflection in the link,  $\alpha$ . Use the parameters shown in Figure 2.2.
3. Compute the Lagrangian of the system.
4. Find the first Euler-Lagrange equation given in 2.4. Keep the equations in terms of applied torque,  $\tau$  (i.e., not in terms of DC motor voltage). Also make sure your equations follow the general form:  $J\ddot{x} + B\dot{x} + Kx = u$ .
5. Find the second Euler-Lagrange Equation 2.5.
6. Find the equations of motion:  $\ddot{\theta} = f_1(\theta, \dot{\theta}, \alpha, \dot{\alpha}, \tau)$  and  $\ddot{\alpha} = f_2(\theta, \dot{\theta}, \alpha, \dot{\alpha}, \tau)$ . Assume the viscous damping of the link is negligible, i.e.,  $B_l = 0$ .
7. Given state  $x$  defined in Equation 2.16, find the linear state-space matrices  $A$  and  $B$ .
8. Find the the natural frequency of the response shown in Figure 2.3 if the peak values for the first and fifth oscillation are:  $t_1 = 1.12$  and  $t_5 = 1.71$  seconds. Because the damping is low, assume the damped and undamped natural frequency are equivalent.

## 2.3 Lab Experiments

In the first part of this laboratory, the stiffness of the flexible link is determined by measuring its natural frequency. In the second part, the state-space model is finalized and validated against actual measurements.

### 2.3.1 Finding Stiffness

In Section 2.1.5 we found an equation describing the free-oscillation response of a second-order system. This can also be used to describe the response of the flexible link when initially perturbed and left to decay.

#### Physical Parameters for the Lab

In order to do some of the laboratory exercises, you will need these values:

$$\begin{aligned}B_{eq} &= 0.004 \text{ N} \cdot \text{m} / (\text{rad/s}) \\J_{eq} &= 2.08 \times 10^{-3} \text{ kg} \cdot \text{m}^2 \\m_l &= 0.065 \text{ kg} \\L_l &= 0.419 \text{ m}\end{aligned}$$

**Note:** The equivalent viscous damping,  $B_{eq}$ , and moment of inertia,  $J_{eq}$ , parameters are for the SRV02 when there is no load (i.e., the parameter found in the SRV02 Modeling Laboratory was for servo with the disc load).

#### Experimental Setup

The *FLEXGAGE Free Response* VI shown in Figure 2.4 is used to find the natural frequency of the flexible link. The VI interfaces with the strain gage sensor of the system and outputs the deflection angle of the link when run. The corresponding measurement is saved and plotted in the *Measure* tab to take measurements.

**IMPORTANT:** Before you can conduct this experiment, you need to make sure that the lab files are configured according to your system setup. If they have not been configured already, then you need to go to Section 4.2 to configure the lab files first.

1. In *Rotary Flexible Link (Student).lvproj*, open the *FLEXGAGE Free Response* VI. The VI front panel is shown in Figure 2.4. Make sure the VI is configured for your data acquisition device, as explained in Section 4.2.
2. Hold on to the SRV02 load gear, click on the run button, and then immediately perturb the flexible link. Keep holding the SRV02 base until the controller is done collecting the data. The *Flexible Link (deg)* scope should be reading a response similarly as shown in Figure 2.4. The VI is set to run for 5 seconds.
3. Attach a plot of the free-oscillation response you obtained. One option is to right-click on the Chart and use the *Export | Export Simplified Image* command.
4. Find the natural frequency of the link. As in Pre-Lab Question 8, the damping is low. Therefore, assume the damped natural frequency (which is being measured) is equivalent to the undamped natural frequency. Use the cursors on the graph in the *Measure* tab to take your measurements (measured link data is saved to this after the VI is stopped).
5. Calculate the stiffness of the flexible link,  $K_s$ .  
**Hint:** Assume the link is a rod when finding the moment of inertia of the link,  $J_l$ .

### 2.3.2 State-Space Model

Create the state-space model representing the Quanser Flexible Link system. This model can then be loaded and used for model validation and control design.

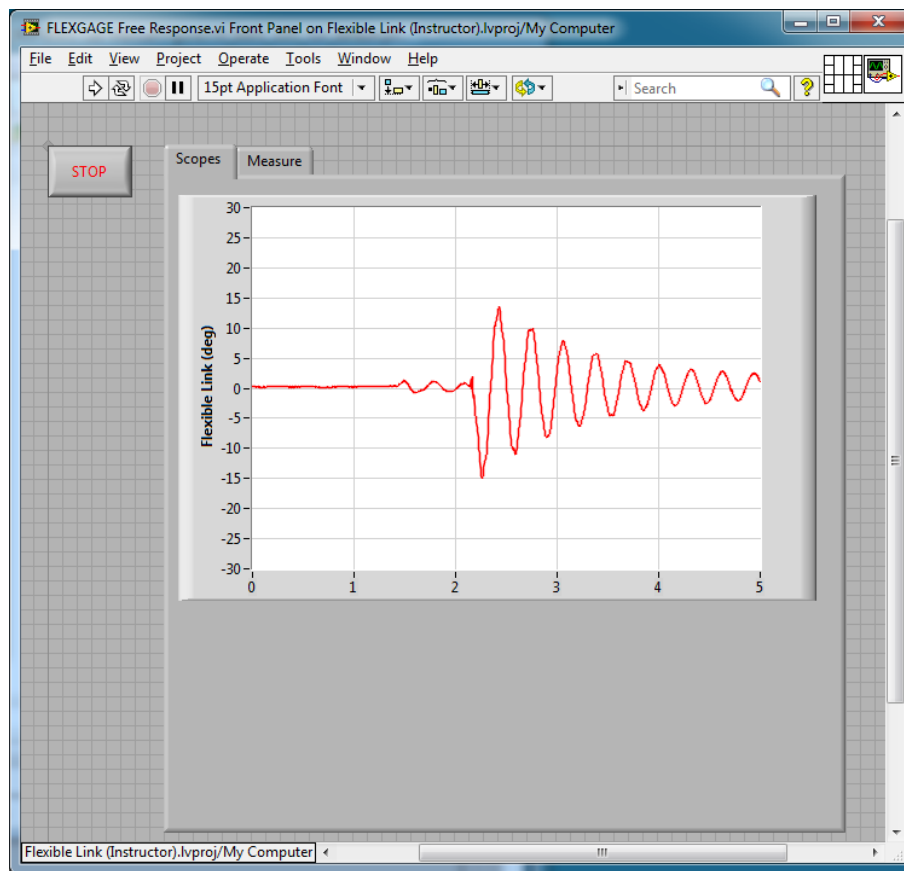


Figure 2.4: Free Response VI used to find link stiffness

### Experimental Setup

The *FLEXGAGE Modeling VI* is shown running with default parameters in Figure 2.5. It is used to generate the state-space model of the FLEXGAGE based on the entered model parameters.

1. In the *Flexible Link (Student).lvproj*, open the *FLEXGAGE Modeling (Student).vi* found in the *Control Design and Simulation* folder. The front panel of the VI is depicted in Figure 2.5.
2. Enter the moment of inertia you calculated in Step 5 of Section 2.3.1 in the *Jl* control box on the front panel of the VI.
3. Enter the stiffness you found in Section 2.3.1 in the *Ks* control box in the front panel.
4. Go to the VI block diagram. The MathScript node has the following state-space matrices entered:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0; \\ 0 & 0 & 0 & 1; \\ 0 & 500 & -5 & 0; \\ 0 & -750 & 5 & 0 \end{bmatrix};$$

$$B = [0 \ 0 \ 500 \ -500]^T;$$

$$C = \text{zeros}(2,4);$$

$$D = \text{zeros}(2,1);$$

5. Enter the state-space matrices you found in Section 2.2 for *A*, *B*, *C*, and *D*. Make sure you use the model parameter inputs that are defined, e.g., the stiffness and link moment of inertia are *Ks* and *Jl* and the SRV02 inertia and damping are denoted *Jeq* and *Beq*.

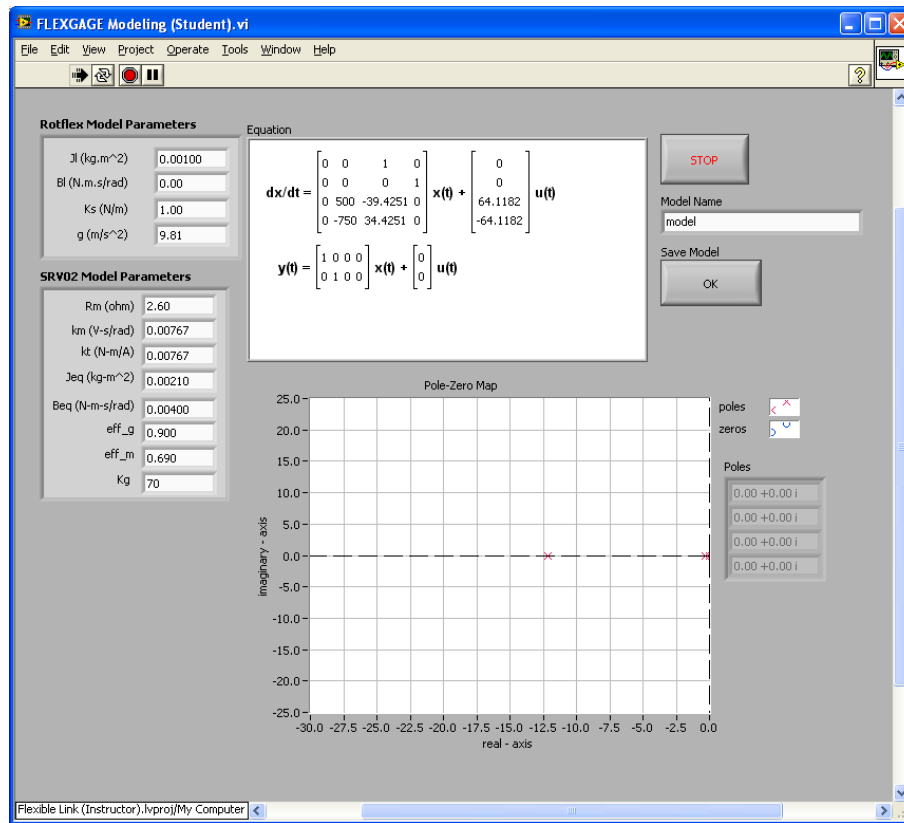


Figure 2.5: FLEXGAGE Modeling VI used to generate state-space model

**Note:** The input of the state-space model you found in Section 2.2 is the torque acting at the servo load gear (i.e., at the pivot of the flexible link). However, *we do not control torque directly - we control the servo input voltage*. Recall the voltage-torque relationship given in Equation 2.9 in Section 2.1.2. At the end of MathScript, the actuator dynamics are added to your state-space matrices with the code:

```
Ao = A;
Bo = B;
B = eta_g*Kg*eta_m*kt/Rm*Bo;
A(3,3) = Ao(3,3) - Bo(3)*eta_g*Kg^2*eta_m*kt*km/Rm;
A(4,3) = Ao(4,3) - Bo(4)*eta_g*Kg^2*eta_m*kt*km/Rm;
```

- Run the VI. It should display a different state-space model. Set the *Model Name* and click on the *Save Model* OK button to save the model on your system. Give the numerical matrices that are shown in the *Equation* display.
- Complete the block diagram shown in Figure 2.6 to display the open-loop poles of the system in the *OL Pole-Zero Map* and *Poles* indicators (using the state-space model). Look through the *Control Design* palette and then run the VI to display the poles.

**Note:** These will be required for a pre-lab question in Section 3.3.

- Now that your model is saved and the open-loop poles are recorded, you can close this VI.

### 2.3.3 Model Validation

By running a simulation and the actual device in parallel, we can verify whether the dynamic model (which drives the simulation) accurately represents our system.

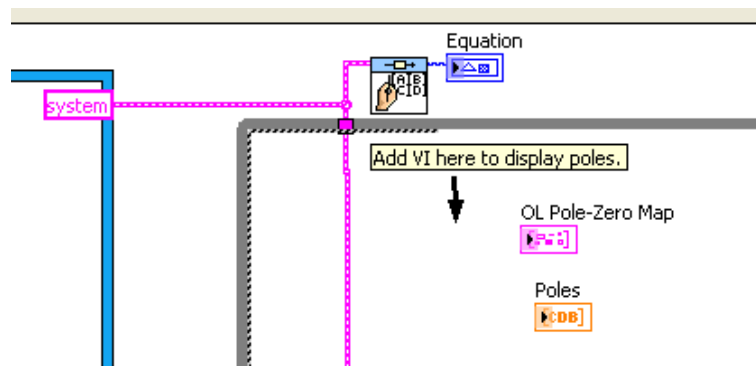


Figure 2.6: Complete block diagram to display poles.

## Experimental Setup

The *FLEXGAGE Model Validation VI* shown in Figure 2.7 applies a step input to both the Quanser Flexible Link hardware and to the Flexible Link model and reads the servo and link angles. The VI interfaces to the actual hardware and outputs the measured servo and link angles. It also reads the  $A$ ,  $B$ ,  $C$ , and  $D$  state-space matrices that are loaded and outputs simulated the servo and link angles.

**IMPORTANT:** Before you can conduct this experiment, you need to make sure that the lab files are configured according to your system setup. If they have not been configured already, then go to Section 4.2 to configure the lab files first.

1. In the *Flexible Link (Student).lvproj*, open the *FLEXGAGE Model Validation (Student)* VI. Make sure the VI is configured for your data acquisition device, as explained in Section 4.2.
2. Make sure the area around the Flexible Link experiment is clear.
3. In the VI block diagram (CTRL-E), double-click on the State-Space block and click on the 'Open' icon to load the model you saved previously from Section 2.3.2.
4. Run the VI. The VI is shown running with the default model given in Figure 2.7. The *Servo Position (deg)* scope displays the simulated servo angle in blue and the measured angle in red. Similarly, the *Flexible Link (deg)* scope shows the simulated flexible link angle in blue and the measured angle in red.
5. If your simulation and measured response match, go to the next step. If they do not, then there is an issue with your model. Here are some items to investigate:
  - Ensure the state-space model was entered properly in the script.
  - The stiffness,  $K_s$ , found in Section 2.3.1 is not correct. Review your calculations or redo the experiment.
  - Review your model derivation in Section 2.2, e.g., there might be a mistake in solving the EOMs.
6. Plot the servo angle, link angle, and input voltage responses and attach them to your report. To save the plots, you can right-click on the chart and go to the *Export | Export Simplified Image with BMP and Clipboard* options.
7. How well does your model represent the actual system? We want a model that is fairly representative but, having said that, keep in mind that no model is perfect. This is just a quick test to see how well your model represents the actual device.
8. Give at least one reason why the model does not represent the system accurately.

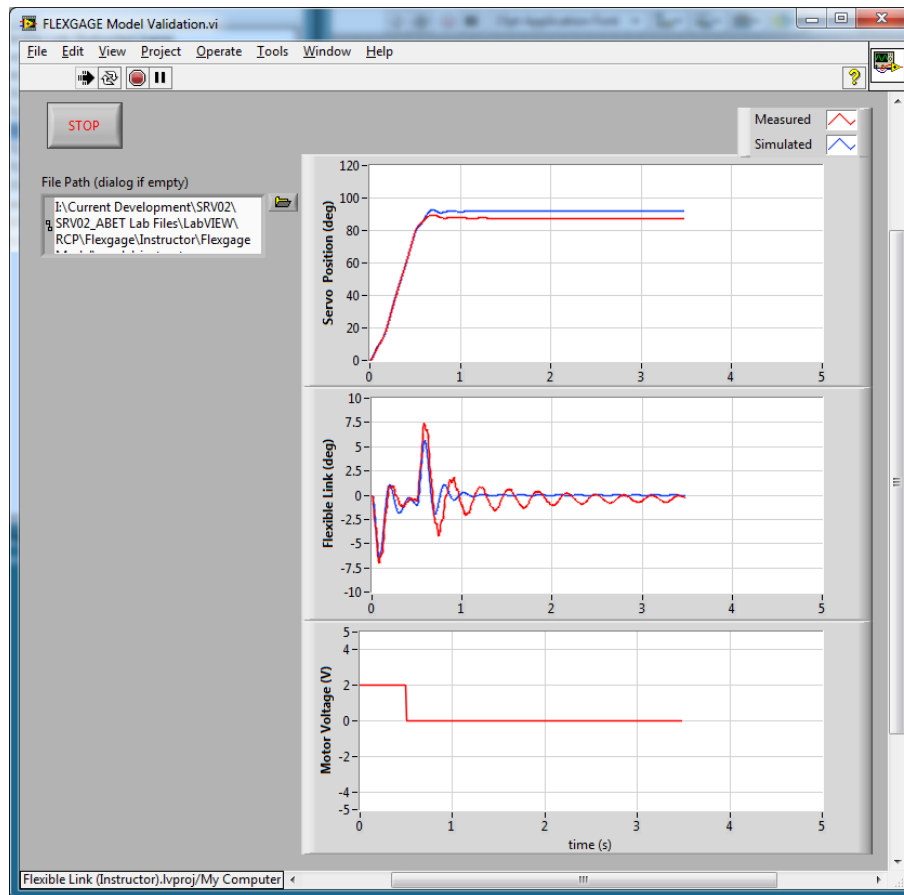


Figure 2.7: VI used validate the model

## 2.4 Results

Fill out Table 2.1 with your answers from your modeling lab results - both simulation and implementation.

Description	Symbol	Value	Unit
<b>Finding Stiffness</b>			
Natural frequency	$\omega_n$		rad/s
Stiffness	$K_s$		N m/rad
<b>Model Validation</b>			
State-Space Matrix	A		
State-Space Matrix	B		
State-Space Matrix	C		
State-Space Matrix	D		
Open-loop poles	OL		

Table 2.1: Results



# 3 CONTROL DESIGN

## 3.1 Specifications

The time-domain requirements are:

**Specification 1:** Servo angle settling time:  $t_s \leq 0.5$  s.

**Specification 2:** Servo angle percent overshoot:  $PO \leq 7.5$  %.

**Specification 3:** Maximum link angle deflection:  $|\alpha| \leq 10$  deg.

**Specification 4:** Maximum control effort / voltage:  $|V_m| \leq 10$  V.

These specifications are to be satisfied when the rotary arm is tracking a  $\pm 30$  degree angle square wave.

## 3.2 Background

In Section 2.2, we found a linear state-state space model that represents the Rotary Flexible Link system. This model is used to investigate the stability properties of the Flexible Link system in Section 3.2.1. In Section 3.2.2, the notion of controllability is introduced. Using the Linear Quadratic Regular algorithm, or LQR, is a common way to find the control gain and is discussed in Section 3.2.3. Lastly, Section 3.2.4 describes the state-feedback control used to control the servo position while minimizing link deflection.

### 3.2.1 Stability

The stability of a system can be determined from its poles ([6]):

- Stable systems have poles only in the left-hand plane.
- Unstable systems have at least one pole in the right-hand plane and/or poles of multiplicity greater than 1 on the imaginary axis.
- Marginally stable systems have one pole on the imaginary axis and the other poles in the left-hand plane.

The poles are the roots of the system's characteristic equation. From the state-space, the characteristic equation of the system can be found using

$$\det(sI - A) = 0 \quad (3.1)$$

where  $\det()$  is the determinant function,  $s$  is the Laplace operator, and  $I$  the identity matrix. These are the *eigenvalues* of the state-space matrix  $A$ .

### 3.2.2 Controllability

If the control input,  $u$ , of a system can take each state variable,  $x_i$  where  $i = 1 \dots n$ , from an initial state to a final state then the system is controllable, otherwise it is uncontrollable ([6]).

**Rank Test** The system is controllable if the rank of its controllability matrix

$$T = [B \ AB \ A^2B \ \dots \ A^nB] \quad (3.2)$$

equals the number of states in the system,

$$\text{rank}(T) = n. \quad (3.3)$$

### 3.2.3 Linear Quadratic Regular (LQR)

If (A,B) are controllable, then the Linear Quadratic Regular optimization method can be used to find a feedback control gain. Given the plant model in Equation 2.14, find a control input  $u$  that minimizes the cost function

$$J = \int_0^{\infty} x(t)' Q x(t) + u(t)' R u(t) dt, \quad (3.4)$$

where  $Q$  and  $R$  are the weighting matrices. The weighting matrices affect how LQR minimizes the function and are, essentially, tuning variables.

Given the control law  $u = -Kx$ , the state-space in Equation 2.14 becomes

$$\begin{aligned} \dot{x} &= Ax + B(-Kx) \\ &= (A - BK)x \end{aligned}$$

### 3.2.4 Feedback Control

The feedback control loop that in Figure 3.1 is designed to stabilize the servo to a desired position,  $\theta_d$ , while minimizing the deflection of the flexible link.

The reference state is defined

$$x_d = [\theta_d \ 0 \ 0 \ 0] \quad (3.5)$$

and the controller is

$$u = K(x_d - x). \quad (3.6)$$

Note that if  $x_d = 0$  then  $u = -Kx$ , which is the control used in the LQR algorithm.

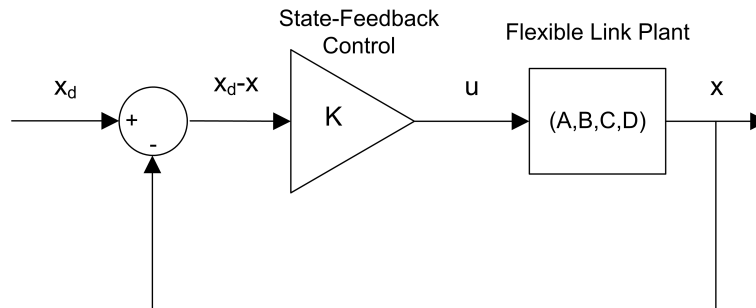


Figure 3.1: State-feedback control loop

### 3.3 Pre-Lab Questions

1. Based on your analysis of the system in the Modeling Laboratory (Section 2.3), is the system stable, marginally stable, or unstable? From your experience in Section 2.3, does the stability you determined analytically match how the actual system behaves?
2. Designing a controller with the Linear Quadratic Regular (LQR) technique is an iterative process. In software, you have to tune the  $Q$  and  $R$  matrices, generate the gain  $K$  using LQR, and either simulate the system or implement the control to see if you have the desired response. The relationship between changing  $Q$  and  $R$  and the closed-loop response is not evident. However, we can have a better idea on how changing the different elements in  $Q$  and  $R$  will affect the response. We will only be changing the diagonal elements in  $Q$ , thus let

$$Q = \begin{bmatrix} q_1 & 0 & 0 & 0 \\ 0 & q_2 & 0 & 0 \\ 0 & 0 & q_3 & 0 \\ 0 & 0 & 0 & q_4 \end{bmatrix}. \quad (3.7)$$

Since we are dealing with a single-input system,  $R$  is a scalar value. Using the  $Q$  and  $R$  defined, expand the cost function given in Equation 3.4.

3. For the feedback control  $u = -Kx$ , the Linear-Quadratic Regular algorithm finds a gain  $K$  that minimizes the cost function  $J$ . Matrix  $Q$  sets the weight on the states and determines how  $u$  will minimize  $J$  (and hence how it generates gain  $K$ ). From your solution in Question 2, explain how increasing the diagonal elements,  $q_i$ , effects the generated gain  $K = [k_1 \ k_2 \ k_3 \ k_4]$ .
4. Explain the effect of increasing  $R$  on the generated gain,  $K$ .

## 3.4 Lab Experiments

The control gain is designed using LQR through simulation first. Once a gain that satisfies the requirements is found, it is implemented on the actual Quanser Flexible Link system.

### 3.4.1 Control Design

The LQR algorithm discussed in Section 3 is implemented in software to find the control gain that gives the desired closed-loop response. The simulation is performed using the linear state-space model of the system.

#### Experimental Setup

The *FLEXGAGE Control Design* VI shown in Figure 3.2 is used to design the feedback control gain using the LQR algorithm. It also simulates the closed-loop response of the Flexible Link using the designed control gain. The VI reads the state-space model from the file that is specified by the user.

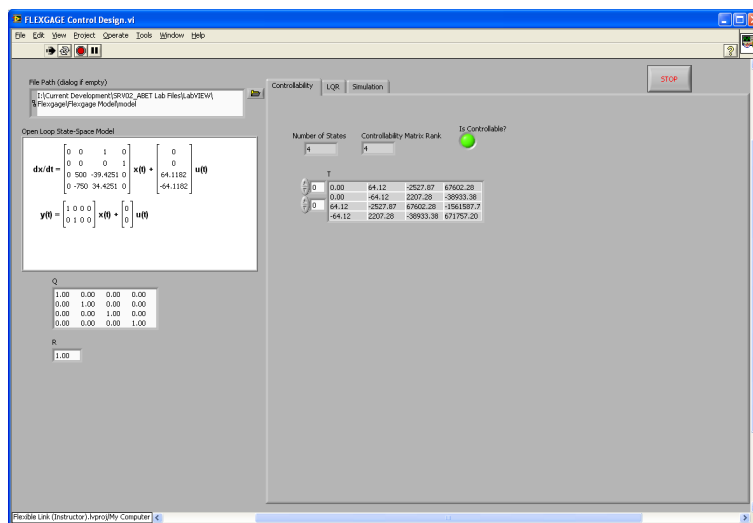


Figure 3.2: VI used to design the state-feedback control gain

**IMPORTANT:** Before you can conduct this experiment, you need to make sure that the lab files are configured according to your system setup. If they have not been configured already, go to Section 4.2 to configure the files first. Make sure the model you found in Section 2.3.3 is in *File Dialog*.

1. In *Rotary Flexible Link (Student).lvproj*, open the *FLEXGAGE Control Design* VI in the *Control Design and Simulation* folder.
2. Run the VI. It should look similar to Figure 3.2 (except using your model). The VI will prompt you to find a model file (unless you selected a model in *File Path* already). Find the model you saved from the previous modeling lab.
3. Go to the *Controllability Test* tab. Is the system controllable? Explain why.
4. Go the *LQR* tab. As depicted in Figure 3.3, this computes a feedback gain  $K$  according to the model you loaded and the  $Q$  and  $R$  matrices that are set in the VI. The  $Q$  and  $R$  are initially set to the default values of:

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } R = 1.$$

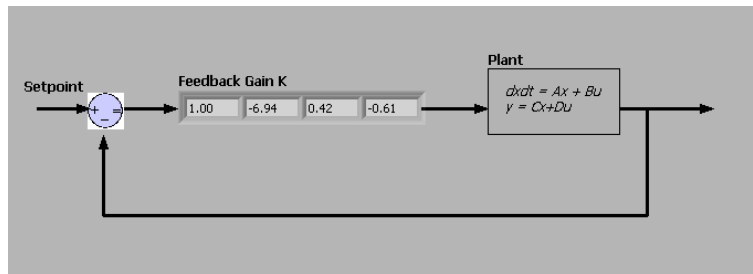


Figure 3.3: Generated LQR Gain

These will not give you the desired response, but generates a default gain  $K$ .

- Go the *Simulation* tab to simulate the closed-loop response with the gain generated back in the *LQR* tab. The simulated response is shown in Figure 3.4

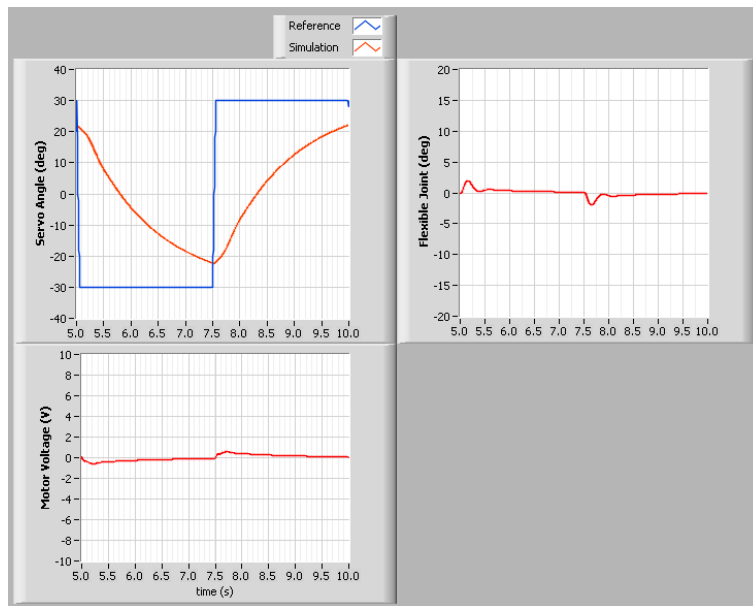


Figure 3.4: Default Simulated Closed-Loop Response

- If  $Q = \text{diag}[q_1, q_2, q_3, q_4]$ , vary each  $q_i$  independently and examine its effect on the gain and the closed-loop response. For example, when increasing  $q_3$ , what happens to  $\theta$  and  $\alpha$ ? Vary each  $q_i$  by the same order of magnitude and compare how the new gain  $K$  changes compared to the original gain. Keep  $R = 1$  throughout your testing. Summarize your results.  
**Note:** Recall your analysis in pre-lab Question 3 where the effect of adjusting  $Q$  on the generated  $K$  was assessed generally by inspecting the cost function equation. You may find some discrepancies in this exercise and the pre-lab questions.
- Find a  $Q$  and  $R$  that will satisfy the specifications given in Section 3.1. When doing this, don't forget to keep the dc motor voltage within  $\pm 10$  V. This control will later be implemented on actual hardware. Therefore, make sure the actuator is not being saturated. Enter the weighting matrices,  $Q$  and  $R$ , used and the resulting gain,  $K$ .
- Briefly explain the procedure you used to find  $Q$  and  $R$ .

### 3.4.2 Control Simulation

Using the linear state-space model of the system and the designed control gain, the closed-loop response can be simulated. This way, we can test the controller and see if it satisfies the given specifications before running it on the

hardware platform.

## Experimental Setup

The *FLEXGAGE Control Simulation VI* shown in Figure 3.5 is used to simulate the closed-loop response of the Flexible Link using the control gain found in Section 3.4.1.

The Signal Generator generates a square wave with an amplitude and frequency specified by the *Amplitude (deg)* and *Frequency (Hz)* controls. These are used to change the desired servo position command. By default, the servo command is a 0.33 Hz square wave of  $\pm 30$  degrees. This is passed through a *Rate Limiter* block to smooth the signal (see the VI block diagram). The state-feedback gain  $K$  is set in the *Feedback Gain* control box on the front panel. The VI loads the state-space model specified by the *File Dialog* box.

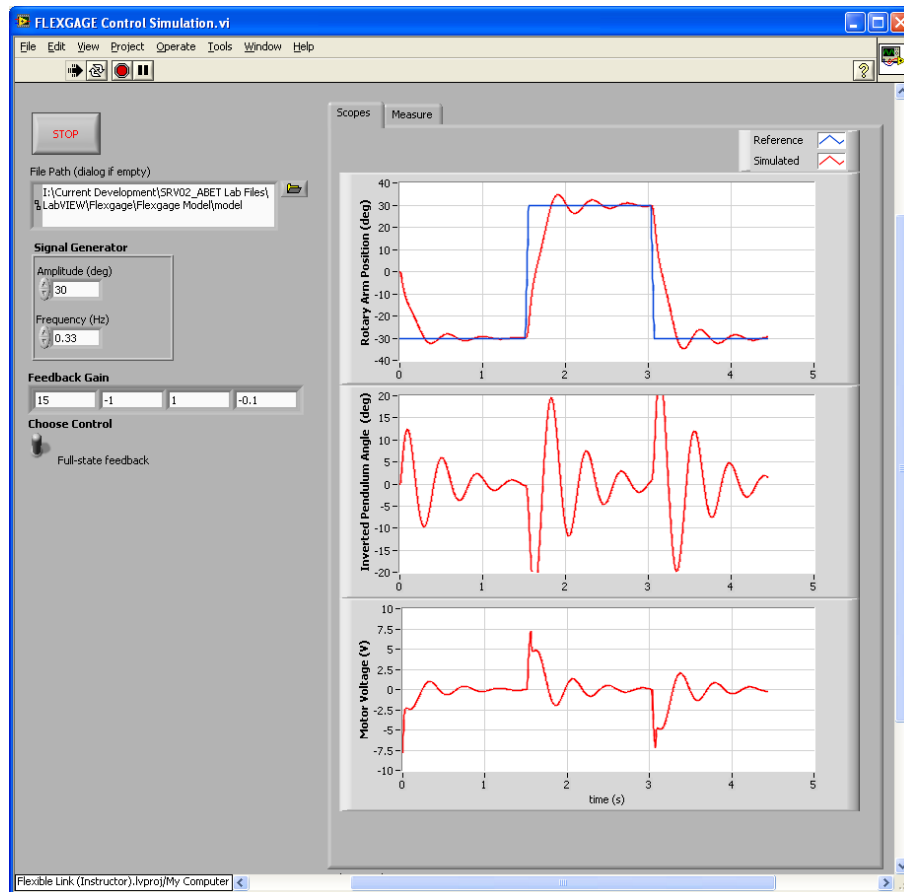


Figure 3.5: VI used to simulate the state-feedback control

**IMPORTANT:** Before you can conduct this experiment, you need to make sure that the lab files are configured according to your system setup. If they have not been configured already, go to Section 4.2 to configure the files first. Make sure the model you found in Section 2.3.3 is in *File Dialog* and the control gain  $K$  you found is entered in the *Feedback Gain* control.

1. In *Rotary Flexible Link (Student).lvproj*, open the *FLEXGAGE Control Simulation VI*.
2. Click the *File Path* control and find the model you saved from Section 2.3.2.
3. Enter the gain  $K$  you found in Section 3.4.1 in *Feedback Gain* control of the VI.
4. Set *Amplitude (deg)* to 30 and *Frequency (Hz)* to 0.33 to generate a  $\pm 30$  degree 0.33 Hz square wave desired servo angle.
5. Make sure the *Choose Control* switch is set to the UPWARD position for *Full-State Feedback* control.

6. Run the VI. It should look similarly as shown in Figure 3.5 (except using your model and gain). The VI will prompt you to find a model file (unless you selected a model in *File Path* already).
7. Stop the VI, by clicking on the *Stop* button, once you have a suitable response.
8. Plot the simulated response of the servo, link, and motor input voltage obtained using your obtained gain  $K$  and attach it to your report.
9. Measure the settling time and percent overshoot of the simulated servo response and the maximum link deflection. Does the response satisfy the specifications given in Section 3.1? Use the cursors in the graphs found in the *Measure* tab to take your measurements.

### 3.4.3 Control Implementation

In this experiment, the servo position is controlled while minimizing the link deflection using the LQR-based control found in Section 3.4.2. Measurements will then be taken to ensure that the specifications are satisfied.

#### Experimental Setup

The *FLEXGAGE Control* VI shown in Figure 3.7 is used to run the state-feedback control on the Quanser Flexible Link system. The VI interfaces with the DC motor and sensors of the system and implements the feedback control.

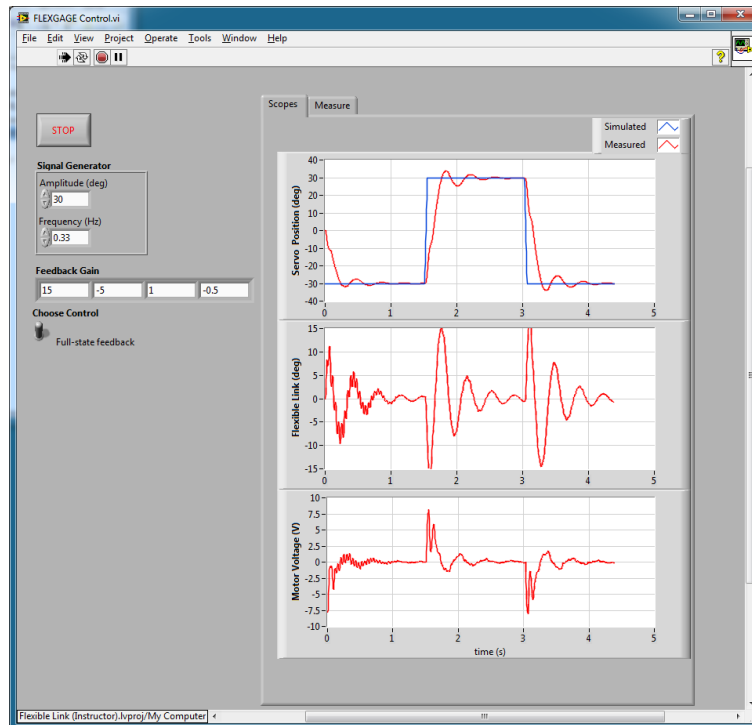


Figure 3.6: VI used to run state-feedback control on Flexible Link system

**IMPORTANT:** Before you can conduct this experiment, you need to make sure that the lab files are configured according to your system setup. If they have not been configured already, then go to Section 4.2 to configure the lab files first.

1. In *Rotary Flexible Link (Student).lvproj*, open the *FLEXGAGE Control* VI. Make sure the VI is configured for your data acquisition device, as explained in Section 4.2.
2. Enter the gain  $K$  you found in Section 3.4.1 in *Feedback Gain* control of the VI.
3. Make sure the *Manual Switch* is set to the upward position for *Full-State Feedback* control.

4. Set *Amplitude (deg)* to 30 and *Frequency (Hz)* to 0.33 to generate a  $\pm 30$  degree 0.33 Hz square wave desired servo angle.
5. Run the VI. The flexible link should be tracking the default  $\pm 30$  degree signal. It should look similar to Figure 3.7 (except using your model and gain).
6. Stop the controller once you have obtained a representative response.
7. Plot the responses from the *theta (deg)*, *alpha (deg)*, and *Vm (V)* scopes.
8. Measure the settling time and percent overshoot of the measured servo response and the maximum link deflection. Does the response satisfy the specifications given in Section 3.1? As with the simulation, use the cursors in the graphs in the *Measure* tab to take your measurements.

### 3.4.4 Implementing Partial-State Feedback Control

In this section, the partial-state feedback response of the system is assessed and compared with the full-state feedback control in Section 3.4.3.

1. As in Section 3.4.3, open the *FLEXGAGE Control* VI (for your DAQ configuration).
2. Ensure the control gain you settled on in Section 3.4.3 is entered in the *Feedback Gain* control.
3. Make sure the *Choose Control* switch is set to the downward position for *Partial-State Feedback* control.
4. Run the VI. The link should be tracking the default  $\pm 30$  degree square wave.
5. Stop the controller once you have obtained a representative response.
6. As in Section 3.4.3, attach plots representing the SRV02 angle and flexible link angle response, as well as the input voltage.
7. Examine the difference between the partial-state feedback (PSF) response and the full-state feedback (FSF) response. Explain why PSF control behaves this way by looking at the *FLEXGAGE Control* VI block diagram.



## 3.5 Results

Fill out Table 3.1 with your answers from your control lab results - both simulation and implementation.

Description	Symbol	Value	Unit
<b>LQR Simulation</b>			
LQR Weighting Matrices	$Q$		
	$R$		
LQR Gain	$K$		
Settling time	$t_s$		s
Percentage overshoot	$PO$		%
Maximum deflection	$ \alpha _{\max}$		deg
<b>LQR Implementation</b>			
LQR Weighting Matrices	$Q$		
	$R$		
LQR Gain	$K$		
Settling time	$t_s$		s
Percentage overshoot	$PO$		%
Maximum deflection	$ \alpha _{\max}$		deg

Table 3.1: Results

# 4 SYSTEM REQUIREMENTS

## Required Hardware

- Data-aquisition (DAQ) device that is compatible with LabVIEW™ and Quanser Rapid Control Prototyping Toolkit®, e.g., NI USB or PCI DAQ, NI CompactRIO, or Quanser Hardware-in-the-loop (HIL).
- Quanser SRV02-ET rotary servo. See Reference [1].
- Quanser Rotary Flexible Link (attached to SRV02).
- Quanser VoltPAQ-X1 power amplifier, or equivalent (e.g. Reference [2] for VoltPAQ User Manual).

## Required Software

- NI LabVIEW™
- NI LabVIEW Control Design and Simulation Module
- NI LabVIEW MathScript RT Module
- For NI CompactRIO users:
  - NI LabVIEW Real-Time Module
  - NI LabVIEW FPGA Module
  - NI-RIO Drivers
- Quanser Rapid Control Prototyping Toolkit®

## 4.1 Overview of Files

File Name	Description
Flexible Link User Manual.pdf	This manual describes the hardware of the Rotary Flexible Link system and explains how to setup and wire the system for the experiments.
Flexible Link Lab Manual (Student).pdf	This laboratory guide contains pre-lab questions and lab experiments demonstrating how to design and implement a position controller on the Quanser SRV02 Flexible Link plant using LabVIEW™.
Flexible Link (Student).lvproj	LabVIEW project containing all the VIs the student needs to complete the labs.
FLEXGAGE Modeling (Student).vi	Generates the state-space model of FLEXGAGE system. Note: VI is incomplete.
FLEXGAGE Free Response.vi	This VI measures the corresponding Flexible Link angle. By manually perturbing the link, the measured response can be used to find the natural frequency of the link.
FLEXGAGE Modeling Validation.vi	VI used to compare the Flexible Link state-space model with the measured response from the actual system.
FLEXGAGE Control Design.vi	Finds the feedback control gain $K$ using Linear Quadratic-Regulator algorithm.
FLEXGAGE Control Simulation.vi	Simulates the Flexible Link system when using a full or partial state-feedback control.
FLEXGAGE Control.vi	Implements the closed-loop state-feedback controller on the Quanser Rotary Flexible Link system using LabVIEW™. Full or partial-state feedback control can be selected.

Table 4.1: Files supplied with the SRV02 Flexible Link Control Laboratory.

## 4.2 Software Setup

Follow these steps to get the system ready for this lab:

1. Load the LabVIEW™ software.
2. Open the LabVIEW project called *Flexible Link (Student).lvproj* shown in Figure 4.1.
3. **Control Design and Simulation Labs:** To run a control design or simulation based VI, go to the *Control Design and Simulation* folder and open the VI that corresponds to the lab.
4. **Finding Stiffness Lab:** To find the natural frequency of the Flexible Link as in Section 2.3.1, open the *FLEXGAGE Free Response* VI.
5. **Model Validation Lab:** When going through the lab explained in Section 2.3.3 to validate your model, open the *FLEXGAGE Model Validation* VI.
6. **Control Implementation Lab:** When going through the control implementation lab in Section 3.4.3 or Section 3.4.4, open the *FLEXGAGE Control* VI.
7. **Configure DAQ:** Before running the VI, make sure you set the correct *Board type* (e.g., 'q1\_cRIO', 'q2\_usb', 'q8\_usb', 'qpid', or 'qpid\_e') in the HIL Initialize block.
8. **Quanser CompactRIO Users:** Before running the VI, make sure you can connect to your CompactRIO through the Measurement & Automation software. See the SRV02 cRIO User Manual ([3]).

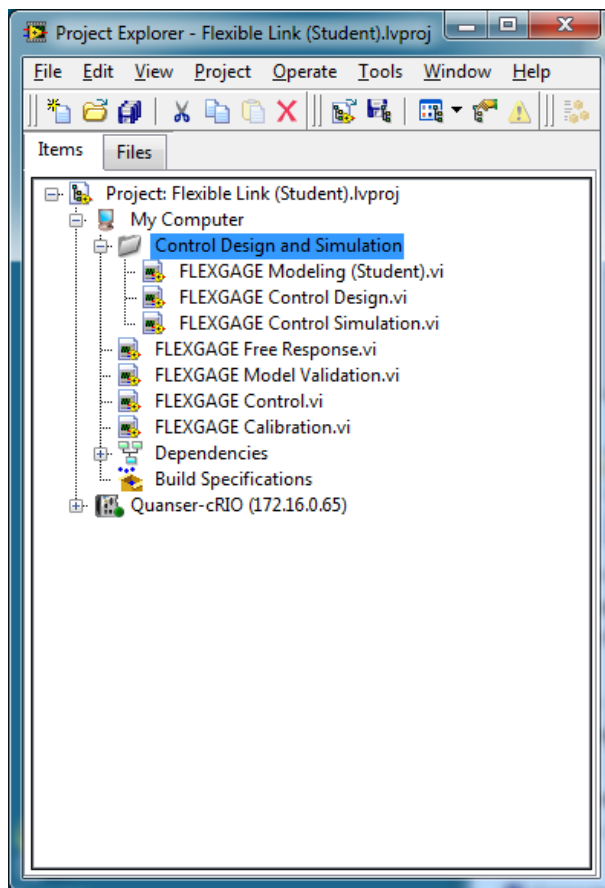


Figure 4.1: Flexible Link LabVIEW Project (Student version).

9. **Channel Configuration:** For any of the DAQ-based VIs, the encoder input and output channels are set, by default, to match the wiring in the Flexible Link User Manual ([5]). If the wiring is different on your system, make sure the VI uses the correct channels. For instance, if your link strain gage is connected to Analog Input Channel #2 on your DAQ, then configure the HIL Analog Read VI channel to read AI #2.

# 5 LAB REPORT

This laboratory contains two groups of experiments, namely,

1. Modeling the Quanser Rotary Flexible Link system, and
2. State-feedback control using LQR.

For each experiment, follow the outline corresponding to that experiment to build the *content* of your report. Also, in Section 5.3 you can find some basic tips for the *format* of your report.

## 5.1 Template for Content (Modeling)

### I. PROCEDURE

#### 1. *Finding Stiffness*

- Briefly describe the main goal of the experiment.
- Briefly describe the experiment procedure (Section 2.3.1)

#### 2. *Model Validation*

- Briefly describe the main goal of the experiment.
- Briefly describe the experiment procedure (Section 2.3.3)

### II. RESULTS

Do not interpret or analyze the data in this section. Just provide the results.

1. Free-oscillation plot from step 3 in Section 2.3.1.
2. Model validation plot from step 6 in Section 2.3.3.
3. Provide applicable data collected in this laboratory (from Table 2.1).

### III. ANALYSIS

Provide details of your calculations (methods used) for analysis for each of the following:

1. Measured link stiffness in step 5 in Section 2.3.1.
2. Model discrepancies given in step 8 in Section 2.3.3.

### IV. CONCLUSIONS

Interpret your results to arrive at logical conclusions for the following:

1. How does the model compare with the actual system in step 7 of Section 2.3.3, *State-space model validation*.

## 5.2 Template for Content (Control)

### I. PROCEDURE

#### 1. Control Design

- Briefly describe the main goal of the control design.
- Briefly describe the control design procedure (Section 3.4.1).
- Briefly describe the procedure in step 6 of Section 3.4.2 to examine the effect of variables on the gain and closed-loop response.
- Briefly explain the procedure used to find  $Q$  and  $R$  in step 8 of Section 3.4.2).

#### 2. Control Simulation

- Briefly describe the main goal of the simulation.
- Briefly describe the simulation procedure (Section 3.4.2).

#### 3. Full-State Feedback Implementation

- Briefly describe the main goal of this experiment.
- Briefly describe the experimental procedure (Section 3.4.3).

#### 4. Partial-State Feedback Implementation

- Briefly describe the main goal of this experiment.
- Briefly describe the experimental procedure (Section 3.4.4).

### II. RESULTS

Do not interpret or analyze the data in this section. Just provide the results.

1. Response plot from step 8 in Section 3.4.2, *Full-state feedback LQR controller simulation*.
2. Response plot from step 7 in Section 3.4.3, for *Full-state feedback LQR controller implementation*.
3. Response plot from step 6 in Section 3.4.4, for *Partial-state feedback LQR controller implementation*.
4. Provide applicable data collected in this laboratory (from Table 3.1).

### III. ANALYSIS

Provide details of your calculations (methods used) for analysis for each of the following:

1. Controllability of system in Step 3 in Section 3.4.1.
2. Settling time and percent overshoot in step 9 in Section 3.4.2, *Full-state feedback LQR controller simulation*.
3. Settling time and percent overshoot in step 8 in Section 3.4.3, for *Full-state feedback LQR controller implementation*.
4. Comparison between partial-state and full-state feedback in step 7 in Section 3.4.4.

### IV. CONCLUSIONS

Interpret your results to arrive at logical conclusions for the following:

1. Whether the controller meets the specifications in step 9 in Section 3.4.2, *Full-state feedback LQR controller simulation*.
2. Whether the controller meets the specifications in step 8 in Section 3.4.3, for *Full-state feedback LQR controller implementation*.

## 5.3 Tips for Report Format

### PROFESSIONAL APPEARANCE

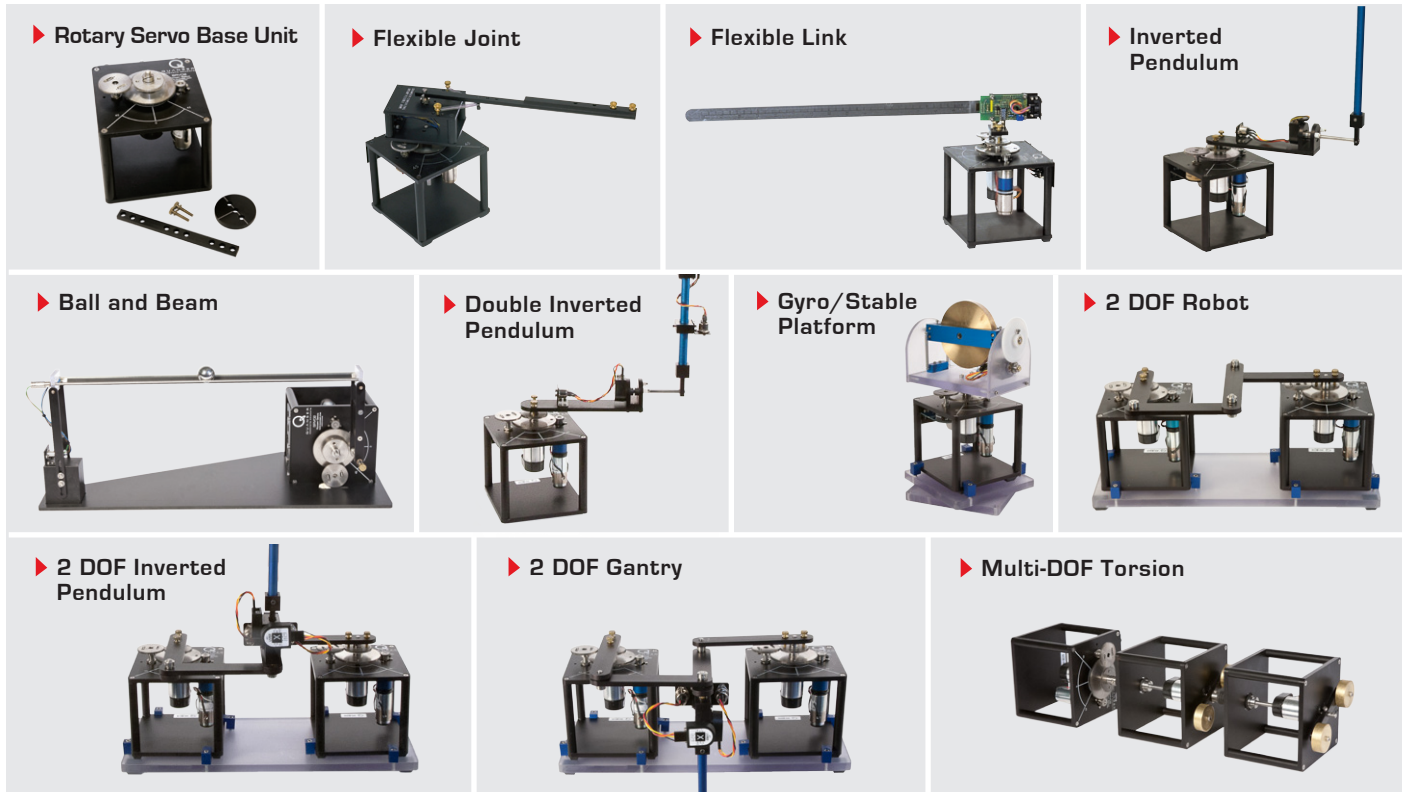
- Has cover page with all necessary details (title, course, student name(s), etc.)
- Each of the required sections is completed (Procedure, Results, Analysis and Conclusions).
- Typed.
- All grammar/spelling correct.
- Report layout is neat.
- Does not exceed specified maximum page limit, if any.
- Pages are numbered.
- Equations are consecutively numbered.
- Figures are numbered, axes have labels, each figure has a descriptive caption.
- Tables are numbered, they include labels, each table has a descriptive caption.
- Data are presented in a useful format (graphs, numerical, table, charts, diagrams).
- No hand drawn sketches/diagrams.
- References are cited using correct format.

# REFERENCES

- [1] Quanser Inc. *SRV02 User Manual*, 2009.
- [2] Quanser Inc. *VoltPAQ User Guide*, 2010.
- [3] Quanser Inc. *SRV02 cRIO User Manual*, 2011.
- [4] Quanser Inc. *SRV02 lab manual*. 2011.
- [5] Quanser Inc. *SRV02 Rotary Flexible Link User Manual*, 2011.
- [6] Norman S. Nise. *Control Systems Engineering*. John Wiley & Sons, Inc., 2008.



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